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VECTOR CALCULUS

1 Introduction

Vector calculus is a branch of mathematics that allows differentiation and integration of (scalar) functions and vector function in several variables at once. It is based upon multivariable calculus. A (scalar) function is a scalar whose value depends upon several variables. Examples are the temperature and pressure of the atmosphere, the density of an inhomogeneous solid and so on. A vector function (also called a vector field) is a vector whose components depend upon several variables. Examples of these are the electric and magnetic field, the velocity of a fluid, and so on.

In the present case we are dealing with vectors in three-dimensional space so they have three components. The number of variables that functions and vector components can depend on is also three.

In this chapter we review the formalism of the nabla operator (∇) and what it is used for in vector calculus.

2 The ∇ Operator

We obviously must require $r \neq 0$.

The fundamental operator we deal with in vector calculus is the ∇ operator. It is defined as

$$\boldsymbol{\nabla} = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{1}$$

The vectors e_x, e_y, e_z are three mutually perpendicular unit vectors that form a right-handed triplet with e_x along the positive x-axis and e_y along the positive y-axis and e_z along the positive z-axis. We do not use the notation i, j, k for these. The partial differentiations are familiar from multivariable calculus. It is seen that the ∇ operator is a vector because it is the sum of three terms that are each a vector. The order of the factors in each term does not matter because the unit vectors are constant vectors, independent of position x, y, z so the partial differentiations will do nothing to them. This is all you need to know (besides some nomenclature) because everything in vector calculus we need follows from the definition (1) of the ∇ operator.

3 The Gradient of a Scalar Function

The gradient of a (scalar) function f(x,y,z), defined as ∇f , is evaluated as

$$\nabla f = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \right) f(x, y, z)$$
(2)

Working out the parentheses in (2) we get

$$\nabla f = \left(\boldsymbol{e}_x \frac{\partial f}{\partial x} + \boldsymbol{e}_y \frac{\partial f}{\partial y} + \boldsymbol{e}_z \frac{\partial f}{\partial z} \right)$$
(3)

It is seen that the gradient of a (scalar) function is a vector.

4 The Divergence of a Vector Function

The divergence of a vector function $\boldsymbol{v}(x, y, z)$, defined as $\boldsymbol{\nabla} \cdot \boldsymbol{v}$, is evaluated as

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot \left(v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z\right)$$
(4)

Working out the parentheses in (4) we get nine terms. The three unit vectors e_x, e_y, e_z form a right-handed triplet of mutually perpendicular unit vectors. So for example $e_x \cdot e_x = 1$, $e_x \cdot e_y = 0$, etc. and we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$
(5)

It is seen that the divergence of a vector function is a scalar as expected for the dot product of two vectors.

5 The Rotation of a Vector Function

The rotation of a vector function $\boldsymbol{v}(x, y, z)$, defined as $\boldsymbol{\nabla} \times \boldsymbol{v}$, is evaluated as

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \times \left(v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z\right)$$
(6)

Working out the parentheses we get nine terms. The three unit vectors e_x, e_y, e_z form a right-handed triplet of mutually perpendicular unit vectors. So for example $e_x \times e_x = 0$, $e_x \times e_y = e_z$, $e_x \times e_z = -e_y$, etc. and we get

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_x - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\boldsymbol{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z \tag{7}$$

where we grouped terms according to e_x , e_y , and e_z . This formula is not easy to remember. It is seen that this result can also be obtained be evaluating the determinant

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$
(8)

The form of the determinant is easier to remember. It is seen that the curl of a vector function is a vector as expected for the cross product of two vectors.

6 Examples

When evaluating an expression involving the nabla operator determine ahead of time whether the result should be a scalar or vector entity.

The simplest example of a gradient is ∇r where $r = \sqrt{x^2 + y^2 + z^2}$ is the length of the vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. We get

$$\boldsymbol{\nabla}r = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) (x^2 + y^2 + z^2)^{\frac{1}{2}}$$
(9)

Working out the parentheses in the first factor we get terms proportional to e_x , e_y , and e_z . The first of these can be evaluated as

$$e_x \frac{\partial}{\partial x} \left(x^2 + y^2 + z^2 \right)^{\frac{1}{2}} = e_x \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} 2x = e_x \frac{x}{r}$$
(10)

Obviously we must require $r \neq 0$. Similar expressions hold for the other two terms. This can be seen by an explicit calculation or by replacing xby y and replacing x by z respectively in (10). It is a good idea to pick up these substitution tricks and save time. They also serve as a check if you did the explicit calculation after all. All three expressions have a factor 1/rin common which we take outside parentheses to get

$$\boldsymbol{\nabla}r = \frac{\boldsymbol{e}_x x + \boldsymbol{e}_y y + \boldsymbol{e}_z z}{r} = \frac{\boldsymbol{r}}{r} \tag{11}$$

We expect ∇r to be a vector and it is. The units work out too.

If $f(x, y, z) = r^n$ then the gradient of f is

$$\nabla f = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{\frac{n}{2}}$$
(12)

Working out the parentheses in the first factor we get terms proportional to e_x , e_y , and e_z . The first of these can be evaluated as

$$e_x \frac{\partial}{\partial x} \left(x^2 + y^2 + z^2 \right)^{\frac{n}{2}} = e_x \frac{n}{2} \left(x^2 + y^2 + z^2 \right)^{\frac{n}{2} - 1} 2x = e_x nx r^{n-2}$$
(13)

Similar expressions hold for the other two terms. All three expressions have a factor nr^{n-2} in common which we take outside parentheses to get

$$\nabla f = \nabla (r^n) = (\boldsymbol{e}_x x + \boldsymbol{e}_y y + \boldsymbol{e}_z z) n r^{n-2} = \boldsymbol{r} n r^{n-2}$$
(14)

We expect ∇f to be a vector and indeed it is. Obviously we must require $r \neq 0$ when n < 2. We could have guessed the answer by using

$$\boldsymbol{\nabla}f(r) = \frac{df}{dr} \boldsymbol{\nabla}r \tag{15}$$

which results in (14) as well. It is often true in vector calculus that when you guess a relation it is often correct but it bears checking. This feature is what makes vector calculus useful. The special case n = -1 corresponds to a 1/r potential such as we encounter in electrostatics and gravity. The force is proportional to the negative of the gradient of f. Setting n = -1in (14) we find that the force is proportional to r/r^3 . The force is along rand has a magnitude proportional to $1/r^2$ as expected. The proportionality constant can be positive or negative so the direction of the force can be in the direction of r or in the direction of -r. Check that n = 2 corresponds to a spring force. Other useful results are epressions for $\nabla \cdot r$ and $\nabla \times r$. Using (5) with $\boldsymbol{v} = \boldsymbol{r} = (x, y, z)$ we find

$$\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3 \tag{16}$$

Calculating $\boldsymbol{\nabla} \times \boldsymbol{r}$ we find using (8)

$$\boldsymbol{\nabla} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{e}_{x} & \boldsymbol{e}_{y} & \boldsymbol{e}_{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$
(17)

which is identically zero so

$$\boldsymbol{\nabla} \times \boldsymbol{r} = 0 \tag{18}$$

Next consider products of a scalar function f = f(x, y, z) and a vector function $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$. Examples are $\nabla \cdot (f\boldsymbol{v})$ and $\nabla \times (f\boldsymbol{v})$. In the first case we get

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{v}) = \left(\boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot \left[f(v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z)\right]$$
(19)

Working out the parenthese we get nine terms. The three unit vectors form a right-handed triplet of mutually perpendicular unit vectors. So for example $e_x \cdot e_x = 1$, $e_x \cdot e_y = 0$, etc and we get

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{v}) = \frac{\partial}{\partial x}(fv_x) + \frac{\partial}{\partial y}(fv_y) + \frac{\partial}{\partial z}(fv_z)$$
(20)

We use the chain rule of differentiation to find

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{v}) = \frac{\partial f}{\partial x}v_x + f\frac{\partial v_x}{\partial x} + \frac{\partial f}{\partial y}v_y + f\frac{\partial v_y}{\partial y} + \frac{\partial f}{\partial z}v_z + f\frac{\partial v_z}{\partial z}$$
(21)

We regroup terms in a suggestive order as

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{v}) = \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z + f\frac{\partial v_x}{\partial x} + f\frac{\partial v_y}{\partial y} + f\frac{\partial v_z}{\partial z}$$
(22)

Using (3) and (5) we see that this equation can be written in a compact form as

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{v}) = (\boldsymbol{\nabla} f) \cdot \boldsymbol{v} + f \nabla \cdot \boldsymbol{v}$$
(23)

You will soon be able to guess this result. It looks like the chain rule of differentiation applied to a product. Knowing that the result has to be a scalar this is the only way it can come out.

Next consider the second case $\nabla \times (fv)$.

$$\boldsymbol{\nabla} \times (f\boldsymbol{v}) = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f\boldsymbol{v}_x & f\boldsymbol{v}_y & f\boldsymbol{v}_z \end{vmatrix}$$
(24)

In working out the determinant we get six terms. The first one is

$$\frac{\partial}{\partial y} \left(f v_z \right) = \frac{\partial f}{\partial y} v_z + f \frac{\partial v_z}{\partial y} \tag{25}$$

where we used the chain rule of differentiation. There are no simplifications as before so we must add all two times six terms to get

$$\nabla \times (f\boldsymbol{v}) = \boldsymbol{e}_x \left(\frac{\partial f}{\partial y} v_z + f \frac{\partial v_z}{\partial y} - \frac{\partial f}{\partial z} v_y - f \frac{\partial v_y}{\partial z} \right) - \\ \boldsymbol{e}_y \left(\frac{\partial f}{\partial x} v_z + f \frac{\partial v_z}{\partial x} - \frac{\partial f}{\partial z} v_x - f \frac{\partial v_x}{\partial z} \right) + \\ \boldsymbol{e}_z \left(\frac{\partial f}{\partial x} v_y + f \frac{\partial v_y}{\partial x} - \frac{\partial f}{\partial y} v_x - f \frac{\partial v_x}{\partial y} \right)$$
(26)

We regroup terms in a suggestive order by writing the odd numbered terms first followed by the even numbered terms

$$\nabla \times (f\boldsymbol{v}) = \boldsymbol{e}_{x} \left(\frac{\partial f}{\partial y} v_{z} - \frac{\partial f}{\partial z} v_{y} \right) - \boldsymbol{e}_{y} \left(\frac{\partial f}{\partial x} v_{z} - \frac{\partial f}{\partial z} v_{x} \right) + \\ \boldsymbol{e}_{z} \left(\frac{\partial f}{\partial x} v_{y} - \frac{\partial f}{\partial y} v_{x} \right) + \boldsymbol{e}_{x} \left(f \frac{\partial v_{z}}{\partial y} - f \frac{\partial v_{y}}{\partial z} \right) - \\ \boldsymbol{e}_{y} \left(f \frac{\partial v_{z}}{\partial x} - f \frac{\partial v_{x}}{\partial z} \right) + \boldsymbol{e}_{z} \left(f \frac{\partial v_{y}}{\partial x} - f \frac{\partial v_{x}}{\partial y} \right)$$
(27)

It is seen that the first six terms can be written in the form of the determinant

$$\begin{vmatrix} \mathbf{e}_{x} & \mathbf{e}_{y} & \mathbf{e}_{z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ v_{x} & v_{y} & v_{z} \end{vmatrix}$$
(28)

The second six terms can also be written as a determinant

$$f \begin{vmatrix} \mathbf{e}_{x} & \mathbf{e}_{y} & \mathbf{e}_{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_{x} & v_{y} & v_{z} \end{vmatrix}$$
(29)

Using (3) and (8) we see that (27) can be written in a compact form as

$$\boldsymbol{\nabla} \times (f\boldsymbol{v}) = (\boldsymbol{\nabla} f) \times \boldsymbol{v} + f(\boldsymbol{\nabla} \times \boldsymbol{v})$$
(30)

You will soon be able to guess this result. It looks like the chain rule of differentiation applied to a product. Knowing that the result has to be a vector this is the only way it can come out when you realize that the only operation that ∇ can do on the scalar function f is taking the gradient.

You might wonder whether the order of f and v on the left side of (30) matters. You might think that exchanging them would replace (30) by

$$\boldsymbol{\nabla} \times (\boldsymbol{v}f) = \boldsymbol{v} \times (\boldsymbol{\nabla}f) + (\boldsymbol{\nabla} \times \boldsymbol{v})f$$
(31)

If this were true that would be bad because the left hand side of (30) and (31) are the same while on the right hand side the first term has changed sign while the second term has not. Fortunately (31) is incorrect. This can be seen be tracing the derivation of (30) for $\nabla \times (vf)$. One finds that the first term becomes

$$\begin{vmatrix} \mathbf{e}_{x} & \mathbf{e}_{y} & \mathbf{e}_{z} \\ v_{x} & v_{y} & v_{z} \\ -\frac{\partial f}{\partial x} & -\frac{\partial f}{\partial y} & -\frac{\partial f}{\partial z} \end{vmatrix}$$
(32)

while the second term remains equal to (29), it does not matter whether f is to the left or the right of the determinant. Comparing (28) and (32)we see that the second and third row are exchanged and that three minus signs appeared. We know from the theory of determinants that if two rows are exchanged the value of the determinant changes sign. Also, a prefactor multiplying a determinant can be brought inside the determinant if one applies it to an entire row or column (any row or column will do). These are general properties of determinants but at this stage you can see that these properties are true when recalling the expression for the curl of two vectors in terms of a determinant. Reversing the order of the two vectors in the curl introduces a minus sign and multiplying one of the vectors by a scalar is the same as having a prefactor in from of the curl. Using these two properties of determinants in (32) we see that the determinant becomes equal to the determinant (28). So even though the order of f and v on the left hand side of (30) is irrelevant the order of the factors on the right hand side of (30) is relevant. The equation should be memorized as an equation for $\nabla \times (fv)$ and not for $\nabla \times (\boldsymbol{v}f)$ when guessing it in the manner indicated above.

Next consider a vector function r/r^n . Obviously we must require that $r \neq 0$ if n > 0. The divergence of this vector function can be calculated using (30) with $f = r^{-n}$ and v = r. In (30) we need

$$\nabla f = \nabla r^{-n} = r(-n)r^{-n-2} \tag{33}$$

where we have used (14) and replaced $n \to -n$. We also need ∇r which equals 3 according to (16). Substituting (33) and (16) in (30) we find

$$\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{r}}{r^n}\right) = \frac{\boldsymbol{\nabla} \cdot \boldsymbol{r}}{r^n} + \boldsymbol{r} \cdot \boldsymbol{\nabla} \left(\frac{1}{r^n}\right) = \frac{3}{r^n} + \boldsymbol{r} \cdot \boldsymbol{r}(-n)r^{-n-2} = \frac{3-n}{r^n} \quad (34)$$

In the special case that n = 0 we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3 \tag{35}$$

Another special case is n = 3 and thus the vector function is $\boldsymbol{v} = \boldsymbol{r}/r^3$ and we get

$$\nabla\left(\frac{\mathbf{r}}{r^3}\right) = 0 \tag{36}$$

We similarly can now evaluate $\nabla^2(1/r)$ where $\nabla^2 = \nabla \cdot \nabla$. Using (14) with f = 1/r so n = -1 we find

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\left(\frac{\boldsymbol{r}}{r^3}\right) \tag{37}$$

Applying ∇ to the left and right side of (37) and using (36) we get

$$\boldsymbol{\nabla}^2 \left(\frac{1}{r}\right) = 0 \tag{38}$$

Next consider the curl of \mathbf{r}/r^n . Obviously we must require that $r \neq 0$ for n > 0. It can be evaluated using (30) with $f = r^{-n}$ and $\mathbf{v} = \mathbf{r}$. In (30) we need $\nabla f = \nabla r^{-n}$ which we take from (33) and $\nabla \times \mathbf{r}$ which is zero according to (18). Substituting these results in (30) we find

$$\boldsymbol{\nabla} \times \frac{\boldsymbol{r}}{r^n} = 0 \tag{39}$$

This is true for all n so we have in particular that $\nabla \times \mathbf{r} = 0$ for n = 0 in agreement with (18).

7 Homework

PROBLEM 1

Calculate $\nabla \phi$ if $\phi = \frac{1}{2}kx^2$. Interpret your answer.

PROBLEM 2

Calculate $\nabla \phi$ if $\phi = \frac{1}{2}kr^2$ with $r = xe_x + ye_y + ze_z$, Interpret your answer.

PROBLEM 3

Sometimes the potential energy U is defined by

$$U_B - U_A = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{s} \tag{40}$$

while at other times it is defined by

$$\boldsymbol{F} = -\boldsymbol{\nabla} U \tag{41}$$

Show that these two definitions are equivalent by substituting (41) into (40). Why is there a minus sign in (40) and (41)?

PROBLEM 4

Explain why a necessary condition for the definition of the potential energy U is that

$$\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0 \tag{42}$$

If (42) is true we call ${\pmb F}$ a conservative force. Make a comparison with the condition

$$\oint dS = \oint \frac{dQ}{T} = 0 \tag{43}$$

required for the entropy S to be defined.

PROBLEM 5

Calculate $\nabla \times \mathbf{r}$ with $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

PROBLEM 6

Calculate $\nabla \times r/r^n$ with $r = xe_x + ye_y + ze_z$. What is *n* for the gravitational force?

PROBLEM 7

Calculate $\nabla \cdot \boldsymbol{r}$ with $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$.

PROBLEM 8

Calculate $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$.