H.P. Paar **PHYS 4C: Electrmagnetism** Fall 2015

Quiz # 3 solutions

Solutions by Yury Kiselev

1. (10 points)

   (a) (5 points) Resistance of the copper wire is

   $$R_c = \frac{\rho l}{A} = \frac{4\rho l}{\pi D^2}.$$

   Because conductivity is the reciprocal to resistivity, resistance of the aluminum wire is

   $$R_A = \frac{l/\sigma}{A} = \frac{4l}{\sigma \pi D^2}.$$

   (b) (5 points) Dissipated power is $W = I^2 R = V^2 R/(R + R_c + R_A)^2$.

2. (15 points)

   (a) (5 points) Effective in this context has the same meaning as RMS – root mean square value. $P_{rms} = V_{rms} I_{rms}$, while $V_{rms} = 120\ V$ in the USA. Then, $I_{rms} = P_{rms}/V_{rms} \approx 16.7\ A$.

   (b) (5 points) Power in the case of DC voltage is $P_{DC} = V_{DC}^2/R = 2\ kW$, but the expression is identical to the power in case of AC voltage: $P = V_{rms}^2/R = 2\ kW$, so $V_{DC} = V_{rms} = 120\ V$ would give the same power dissipation.

   (c) (3 points) $I_{max} = \sqrt{2} I_{rms} \approx 23.6\ A$.

   (d) (2 points) $I = I_{max} \sin(2\pi f t + \phi_0)$, where $f = 60$ Hz and ϕ_0 is an arbitrary phase.

3. (20 points)

   (a) (12 points) Let's assign positive direction of the current I_1, flowing through resistor R_1 to be down (in the picture) and positive direction of I_2 will be up. Let's apply Kirchhoff's law to the the left loop, containing R_1 and R_2. We will go counterclockwise, starting just before the battery. Then,

   $$V_1 - I_1 R_1 - I_2 R_2 = 0.$$

   The same law, applied to the right loop, containing R_2 and battery V_2, will give us $V_2 + I_2 R_2 = 0$. From this, $I_2 = -V_2/R_2$ and then from the first equation,

   $$I_1 = (V_1 - I_2 R_2)/R_1 = (V_1 + V_2)/R_1.$$

   Units of a current are amperes.

(b) (8 points)

$$P_1 = I_1^2 R_1 = (V_1 + V_2)^2 / R_1$$

and

$$P_2 = I_2^2 R_2 = V_2^2 / R_2.$$

Units of power are watts.

4. (20 points)

(a) (4 points) Right after the switch is closed, the voltage drop across C_2 is still zero, because the second capacitor is still uncharged. Applying Kirchhoff's law to the loop gives us $V_1 = I_0 R$, so the instantaneous current is $I_0 = V_1/R$.

(b) (4 points) Total charge on the adjacent (top) plates of the first and second capacitors equals to the initial charge $q_0 = C_1 V_1$ due to the conservation of charge. So, using Kirchhoff's law for the loop in the clockwise direction, we get

$$\frac{q_1}{C_1} - \frac{q_2}{C_2} - IR = 0,$$

where q_1 and q_2 are charges on the top plates of the capacitors.
We know that $q_1 + q_2 = q_0$, so $q_2 = q_0 - q_1$, and the current I, directed to the left equals to $I = -\frac{\mathrm{d}q_1}{\mathrm{d}t}$. The equation above, then, becomes

$$\frac{q_1}{C_1} - \frac{(C_1 V_1 - q_1)}{C_2} + R\frac{\mathrm{d}q_1}{\mathrm{d}t} = 0.$$

We can simplify this expression and get

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} + \frac{q_1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{C_1 V_1}{C_2 R}.$$

We can also get differential equation that depends on the current. It's possible to obtain it from the equation above by differentiating. $I = -\frac{\mathrm{d}q_1}{\mathrm{d}t}$, so

$$\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = 0.$$

[This question was formulated to calculate the differential equation(DE), which was supposed to mean write down the DE. In case you solved the DE, you get additional 12 points for this part. The total number of points for problem 4 still can not exceed 20.]
Solution of the DE depending on the current is $I = I_0 e^{-t/\tau}$, where $\tau = RC_1C_2/(C_1 + C_2)$ and I_0 is the initial maximum current, $I_0 = V_1/R$.
Solution of the DE depending on the charge is $q_1 = q_a e^{-t/\tau} + q_f$ with the same τ.
We now that at $t = 0$: $q_1 = C_1 V_1 = q_a + q_f$, and substituting this expression in the DE, we have $q_f/R(1/C_1 + 1/C_2) = C_1 V_1/(C_2 R)$, so
$q_f = C_1^2 V_1/(C_1 + C_2)$, and $q_a = C_1 C_2 V_1/(C_1 + C_2)$.

(c) (4 points) Without the resistor, the charge and current will be oscillating. The resistor provides a way to dissipate the energy, necessary for reaching equilibrium configuration.

(d) (4 points) At $t = \infty$, the current is zero, so the voltage across two capacitors are same in magnitude, but opposite, if we count them going around the loop. $q_1/C_1 = q_2/C_2$. Also, due to the conservation of charge, $q_1 + q_2 = q_1^0 = C_1 V_1$. Solving these two equations, we get $q_1 = C_1^2 V_1/(C_1 + C_2)$ and $q_2 = C_1 C_2 V_1/(C_1 + C_2)$.

(e) (4 points) The total energy should not be conserved – due to the resistor that dissipates energy. Let's check that.

$$E_i = \tfrac{1}{2} C_1 V_1^2,$$

and the final energy is

$$E_f = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2} = \frac{1}{2}\frac{C_1^4 V_1^2}{(C_1+C_2)^2 C_1} + \frac{1}{2}\frac{C_1^2 C_2^2 V_1^2}{(C_1+C_2)^2 C_2} = \frac{1}{2}\frac{C_1^3 V_1^2 + C_1^2 C_2 V_1^2}{(C_1+C_2)^2} = \frac{1}{2}\frac{V_1^2 C_1^2}{C_1+C_2}.$$

So we see that the final energy is different than the initial by factor of $\frac{C_1}{C_1+C_2} < 1$. The difference was dissipated by the resistor as a heat.

5. (35 points)

(a) (5 points) $\vec{B} = \vec{\nabla} \times \vec{A}$, and the Nabla operator is a derivative with the respect to coordinates, so in terms of units. $[B] = (1/m)[A]$, so units of $\vec{A}$, $[A] = T \cdot m$

(b) (15 points) I will show you another trick how to remember exact form of a vector product, besides the determinant trick. The result of $\vec{B} = \vec{\nabla} \times \vec{A}$, for example, can be written as $\vec{B} = \sum_{ijk} \epsilon^{ijk} \vec{e}_i \partial_j A_k$, where $\vec{e}_i$ are the unit vectors in x,y and z directions, $\partial_j A_k = \frac{\partial A_k}{\partial r_j}$ and tensor $\epsilon^{ijk} = 0$ if not all three indices are different; equals to $+1$, if the three indices are a circular ('even') permutation of xyz and equals to -1 if not. To clarify, there are three cirular permutations of (x, y, z): (x, y, z), (y, z, x) and (z, x, y). And there are three 'odd' permutations: (x, z, y), (y, x, z) and (z, y, x). Note that I switched the last two indices of each of the circular permutations to get an odd one. The names odd and even depend on the number of swaps we need to make to get a given permutation from the initial (x, y, z).
So, $\vec{B} = \vec{\nabla} \times \vec{A} = +\vec{e}_x \partial_y A_z + \vec{e}_y \partial_z A_x + \vec{e}_z \partial_x A_y - \vec{e}_x \partial_z A_y - \vec{e}_y \partial_x A_z - \vec{e}_z \partial_y A_x$, signs $+$ and $-$ are placed according to the corresponding permutation of indices. Then, after collecting terms with the same $\vec{e}_i$, we get $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{e}_x(\partial_y A_z - \partial_z A_y) + \vec{e}_y(\partial_z A_x - \partial_x A_z) + \vec{e}_z(\partial_x A_y - \partial_y A_x)$.
After calculating the derivatives, we get

$$\vec{B} = (2Cyz, -2Cxz, 0).$$

(c) (5 points)

$$|\vec{B}| = 2C|z|\sqrt{x^2 + y^2}.$$

(d) (10 points)

$$\vec{r} \cdot \vec{B} = (x, y, z) \cdot (2Cyz, -2Cxz, 0) = 0.$$

The conclusion is that radius-vector $\vec{r}$ and magnetic field $\vec{B}$ are perpendicular at all points. z-component of $\vec{B}$ is zero, so it's a rotationally-symmetric field, going around z-axis.

Could we know that looking at answer in b)? Yes, but it's much easier to just calculate scalar product and see it. That's how we could anticipate that: we can rotate two dimensional radius-vector (x, y) around z-axis by 90 degrees and get $(y, -x)$. This second vector is perpendicular to the original one, and $\vec{B} = 2Cz(y, -x, 0)$ is perpendicular then to $(x, y, 0)$ and also perpendicular to (x, y, z) because the z component is zero.