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Hans P. Paar

DIVERGENCE AND CURL THEOREMS

1 Introduction

We discuss the theorems of Gauss and Stokes also called the divergence and curl theorems. The latter names are derived from the fact that they involve the divergence and curl operators. We will not derive them here, for that you should consult the text assigned to Math 20E. Use is made of two other documents: vectoralgebra.pdf and nablaoperator.pdf

2 Gauss' Theorem

Consider a volume V that is enclosed by a surface S. Unit vectors $\hat{\boldsymbol{n}}$ are normal to the surface and are directed outward. Gauss' Theorem states

$$\oint_V \boldsymbol{\nabla} \cdot \boldsymbol{v} \, dV = \oint_S \boldsymbol{v} \cdot \boldsymbol{dA} \tag{1}$$

where $\boldsymbol{v}(\boldsymbol{r})$ is a vector function, also called a vector field. Here $\boldsymbol{dA} = \hat{\boldsymbol{n}} \, dA$. The vector $\boldsymbol{v}$ may also depend upon other variables such as time but those are irrelevant for Gauss' Theorem. Gauss' Theorem is also called the Divergence Theorem because of the appearance of $\boldsymbol{\nabla}\cdot$.

As an example we consider the case where the surface S is a sphere with radius R and the vector field is the radius vector $\boldsymbol{r}$. For the left hand side of (1) we must evaluate $\boldsymbol{\nabla} \cdot \boldsymbol{r}$. We know that $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$. Thus the integrand of the volume integral has the constant value 3 and can be taken outside the volume integral. The integral over the volume is $4\pi R^3/3$ so the left hand side of (1) is $4\pi R^3$. For the right hand side of (1) we use that $\hat{\boldsymbol{n}} = \boldsymbol{r}/r$ so that $\boldsymbol{v} \cdot \hat{n} = \boldsymbol{r} \cdot \boldsymbol{r}/r = r$. Thus the integrand of the surface integral has the constant value R and can be taken outside the surface integral. The integral over S is $4\pi R^2$ so the right hand side of (1) is $4\pi R^3$. The two sides of (1) are seen to be equal.

3 Stokes' Theorem

Consider a surface S that is bounded by a line s. The direction for the line integral is defined by the direction of unit vectors $\hat{\boldsymbol{\tau}}$ tangent to the curve. Unit vectors $\hat{\boldsymbol{n}}$ are normal to the surface in a direction relative to the direction of the line integral defined by the right-hand rule. This definition is different from the one for Gauss' Theorem because here there is no definition possible for inside or outside the surface. Stokes' Theorem states

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot \boldsymbol{d}A = \oint_s \boldsymbol{v} \cdot d\boldsymbol{\ell} \tag{2}$$

where $\boldsymbol{v}(\boldsymbol{r})$ is a vector function as above. Here $\boldsymbol{d}\ell = \hat{\boldsymbol{\tau}} d\ell$ and as in the previous Section $\boldsymbol{d}A = \hat{\boldsymbol{n}}\; dA$. The vector $\boldsymbol{v}$ may also depend upon other variables such as time but those are irrelevant for Stokes' Theorem. Stokes' Theorem is also called the Curl Theorem because of the appearance of $\boldsymbol{\nabla}\times$.

As an example we consider the case where the line s is a circle of radius R in the x, y plane. Furthermore, let the surface S be the plane inside the circle s. Consider a vector $\boldsymbol{r}$ that has its endpoint laying on the circle so $\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y$ with $x^2 + y^2 = R^2$. Here $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors along the positive x and y axes respectively. The vector $\boldsymbol{s} = y\,\boldsymbol{e}_x - x\,\boldsymbol{e}_y$ is tangential to the circle as can be seen by evaluating $\boldsymbol{s} \cdot \boldsymbol{r}$, which is zero for all values of x and y so $\boldsymbol{s}$ and $\boldsymbol{r}$ are perpendicular everywhere. We now check that (2) is correct when we chose the vector $\boldsymbol{v}$ to be the vector $\boldsymbol{s}$. For the left hand side of (2) we must evaluate $\boldsymbol{\nabla} \times \boldsymbol{s}$. We find

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{s} &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} \\
&= \boldsymbol{e}_x\big[\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-x)\big] - \boldsymbol{e}_y\big[\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}y\big] + \boldsymbol{e}_z\big[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y\big] \\
&= -2\boldsymbol{e}_z
\end{aligned} \tag{3}$$

At the point $(x, y) = (0, R)$ the vector $\boldsymbol{s} = -R\,\boldsymbol{e}_y$. We choose $\hat{\boldsymbol{\tau}}$ parallel to $\boldsymbol{s}$. Note the direction of $\boldsymbol{\tau}$ in the x, y plane. Using the right-hand rule applied to $\boldsymbol{\tau}$ we find that the normal on the plane of the circle is given by

$\hat{\boldsymbol{n}} = -\boldsymbol{e}_z$. Thus $(\boldsymbol{\nabla} \times \boldsymbol{s}) \cdot \hat{\boldsymbol{n}} = -2\boldsymbol{e}_z \cdot (-\boldsymbol{e}_z) = +2$ and the integrand of the surface integral has the constant value 2 and can be taken outside the surface integral. The integral over the surface is πR^2 so the left hand side of (2) is $2\pi R^2$. For the right hand side of (2) we use that $\boldsymbol{s}$ and $\boldsymbol{d\ell}$ are parallel and that the length of the vector $\boldsymbol{s}$ equals R so that $\boldsymbol{s} \cdot \boldsymbol{d\ell} = R\,d\ell$. Thus the integrand of the line integral has the constant value R and can be taken outside the line integral. The integral along s is $2\pi R$ so the right hand side of (2) is $2\pi R^2$. The two sides of (2) are seen to be equal.

4 Differential Form of Maxwell's Equations

Maxwell's equations in integral form are

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{d}A = \frac{1}{\epsilon} \oint \rho\, dV \tag{4}$$

$$\oint_S \boldsymbol{B} \cdot \boldsymbol{d}A = 0 \tag{5}$$

$$\oint_s \boldsymbol{E} \cdot \boldsymbol{d\ell} = -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \boldsymbol{d}A \tag{6}$$

$$\oint_s \boldsymbol{B} \cdot \boldsymbol{d\ell} = \mu \int_S \boldsymbol{j} \cdot \boldsymbol{d}A + \mu\epsilon \frac{\partial}{\partial t} \int_S \boldsymbol{E} \cdot \boldsymbol{d}A \tag{7}$$

They are called to be in integral form" because of the obvious integrals involved in their notation. We will not discuss Maxwell's equations but instead refer to Giancoli. It is usually easier to differentiate than to integrate so we will reformulate Maxwell's eqauations in differential form. Their differential form involves differentiation.

We use Gauss' Theorem to get

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{d}A = \oint \boldsymbol{\nabla} \cdot \boldsymbol{E}\, dV \tag{8}$$

$$\oint_S \boldsymbol{B} \cdot \boldsymbol{d}A = \oint \boldsymbol{\nabla} \cdot \boldsymbol{B}\, dV \tag{9}$$

and Stokes' Theorem to get

$$\oint_s \boldsymbol{E} \cdot \boldsymbol{d\ell} = \oint_S \boldsymbol{\nabla} \times \boldsymbol{E} \cdot \boldsymbol{d}A \tag{10}$$

$$\oint_s \boldsymbol{B} \cdot \boldsymbol{d\ell} = \oint_S \boldsymbol{\nabla} \times \boldsymbol{B} \cdot \boldsymbol{d}A \tag{11}$$

$$\tag{12}$$

Combining (4) with (8), (5) with (9), (6) with (10), and (8) with (11) we get

$$\oint \boldsymbol{\nabla} \cdot \boldsymbol{E}\, dV = \frac{1}{\epsilon} \oint \rho\, dV \tag{13}$$

$$\oint \boldsymbol{\nabla} \cdot \boldsymbol{B}\, dV = 0 \tag{14}$$

$$\oint_S \boldsymbol{\nabla} \times \boldsymbol{E} \cdot \boldsymbol{d}A = -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \boldsymbol{d}A \tag{15}$$

$$\oint_S \boldsymbol{\nabla} \times \boldsymbol{B} \cdot \boldsymbol{d}A = \mu \int_S \boldsymbol{j} \cdot \boldsymbol{d}A + \mu\epsilon \frac{\partial}{\partial t} \int_S \boldsymbol{E} \cdot \boldsymbol{d}A \tag{16}$$

The relations (13) through (16) must hold for arbitrary volumes, surfaces, and lines so in each equation, the integrands must be equal. Thus we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon} \tag{17}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{18}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{19}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{j} + \mu\epsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{20}$$

In vacuum $\rho = \boldsymbol{j} = 0$, $\epsilon = \epsilon_0$, and $\mu = \mu_0$ and Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \tag{21}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{22}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{23}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{24}$$

There is a certain symmetry in these equations. Of particular note is the minus sign in (23), a manifestation of Lenz' law. If this minus sign was not there, or more accurately if (23) and (24) did not have opposite signs, electromagnetic waves would not exist!

5 Scalar Potential

When discussing the electric field in electrostatics it useful to introduce a (scalar) potential V such that

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{25}$$

Substituting this equation for $\boldsymbol{E}$ in (17) we get

$$\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\boldsymbol{\nabla}^2 V = \frac{\rho}{\epsilon} \tag{26}$$

or

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon} \tag{27}$$

This equation is called Poisson's equation. Note that

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{28}$$

The differential operator in (27) and (28) is called the Laplacian after mathematician Laplace, an exceedingly important equation in *many* branches of physics. Equation (27) is a linear, second order, and inhomogeneous differential equation. As is known from the study of differential equations, its solution consists of a linear superposition of two general solutions of the homogeneous equation obtained from (27) by setting its right hand side to zero and a particular solution where the right hand side of (27) is kept. The study of solutions of (27) is a subject of considerable mathematical development and will be left for the another course. We will find the particular solution of (27).

In the case of electrostatics we already know the solution of (27) because we can use Coulonb's Law to obtain it. To obtain the potential $V(\boldsymbol{r})$ at position $\boldsymbol{r}$ we subdivide the charge distribution ρ in very small volumes dV which have charge $\rho(\boldsymbol{r}')\,dV$. The vector $\boldsymbol{r}'$ specifies the location of a small volume. Notice that we use the same letter V for potential and volume.

Because we will let dV go to zero the small volumes of charge can be considered point charges and we can use the expression for the potential of a point charge $\rho\,dV$ at location $\boldsymbol{r}'$ to obtain its contribution to the potential $V(\boldsymbol{r})$

$$dV(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\frac{\rho(\boldsymbol{r}')\,dV}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{29}$$

To obtain $V(\boldsymbol{r})$ of the entire charge distribution we must integrate (29) and get

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\boldsymbol{r}')\, dV}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{30}$$

By construction (30) is the solution of (27).

6 Vector Potential

When discussing the electric field in electrostatics we found it useful to introduce a (scalar) potential V such that $\boldsymbol{E} = -\boldsymbol{\nabla} V$. In the case of a static magnetic field this can not be copied as for example $\boldsymbol{B} = \boldsymbol{\nabla} W$ because $\boldsymbol{B}$ is not a conservative vector field as can be seen from (7). In the static case the derivative with respect to time is zero but the term with the current density $\boldsymbol{j}$ makes the loop integral over $\boldsymbol{B}$ non-zero and therefore the calculation of W path dependent. Compare this with the situation encountered in electrostatics where the derivative with respect to time is zero and (6) shows that the loop integral over $\boldsymbol{E}$ is therefore zero making the calculation of V path independent.

Instead we introduce the vector potential $\boldsymbol{A}$ that is related to the magnetic field by

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{31}$$

That this is useful is evident when one realizes that this expression for $\boldsymbol{B}$ satisfies (18) because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ for any $\boldsymbol{A}$.

An example of a vector potential that describes a constant magnetic field is $\boldsymbol{A} = \frac{1}{2}\boldsymbol{C} \times \boldsymbol{r}$ with C a constant vector. Substitution of this expression into (31) gives

$$\boldsymbol{B} = \tfrac{1}{2}\boldsymbol{\nabla} \times (\boldsymbol{C} \times \boldsymbol{r}) \tag{32}$$

To work out the this expression we use the identity from vector algebra

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c} \tag{33}$$

The nabla operator works only on $\boldsymbol{r}$ and not on $\boldsymbol{C}$ because the latter is constant. Thus we must keep the nabla operator to the left of $\boldsymbol{r}$ but do not care about the position of $\boldsymbol{C}$. We obtain

$$\boldsymbol{\nabla} \times (\boldsymbol{C} \times \boldsymbol{r}) = (\boldsymbol{\nabla} \cdot \boldsymbol{r})\, \boldsymbol{C} - (\boldsymbol{C} \cdot \boldsymbol{\nabla})\, \boldsymbol{r} \tag{34}$$

Now $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$ and $(\boldsymbol{C} \cdot \boldsymbol{\nabla})\,\boldsymbol{r} = \boldsymbol{C}$ so that (34) bcomes $3\boldsymbol{C} - \boldsymbol{C} = 2\boldsymbol{C}$. Substituting this result in (32) we obtain $\boldsymbol{B} = \boldsymbol{C}$, a constant magnetic field of magniture $|\boldsymbol{C}|$. By setting $\boldsymbol{C} = \boldsymbol{B}$ we get the desired magnitude for $\boldsymbol{B}$.

Another useful example is a cylindrically symmetric magnetic field. It can be described by $\boldsymbol{A} = C\boldsymbol{e}_z/\sqrt{(}x^2 + y^2)$ with C a constant scalar. Substitution of this expression for $\boldsymbol{A}$ in (31) shows that the resulting magnetic field is given by $\boldsymbol{B} = -C(y\boldsymbol{e}_x - x\boldsymbol{e}_y)/\boldsymbol{r}^3$, independent of z. $\boldsymbol{B}$ has no z component and is therefore parallel to the x, y plane. The magnitude is given by $|\boldsymbol{B}| = B = C/r^2$ so lines of constant B are circles parallel to the x, y plane. Because B is independent of z, it is constant on cylindrical surfaces whose centerline is the z-axis. The magnetic field $\boldsymbol{B}$ is tangential to the cylindrical surfaces of constant B. This can be seen by calculating $\boldsymbol{r} \cdot \boldsymbol{B}$ to find that it is zero. Therefore $\boldsymbol{r}$ and $\boldsymbol{B}$ are perpendicular to each other, making $\boldsymbol{B}$ tangential as stated. There are still two possible directions. To find which way $\boldsymbol{B}$ is pointing we calculate $\boldsymbol{B}$ at the point $x = r$ and $y = 0$. We find that $\boldsymbol{B} = C\boldsymbol{e}_y/r^2$ so for points on the positive x-axis $\boldsymbol{B}$ is parallel to $\boldsymbol{e}_y$. This means that $\boldsymbol{B}$ is directed in the direction of increasing angle of $\boldsymbol{r}$ with the x-axis. In other words, the direction of $\boldsymbol{B}$ can be found using the right-hand rule with the thumb pointing in the direction of the vector potential $\boldsymbol{A}$. This is indeed a circular magnetic field whose magnitude falls off as $1/r^2$.

There is some freedom in setting the scalar potential: one can add a constant, say C, to it without changing the electric field because $\boldsymbol{\nabla}(V + C) = \boldsymbol{\nabla}V$. Something like that is true for the vector potentila as well. One can add $\boldsymbol{\nabla}\chi$, with χ a scalar function, to $\boldsymbol{A}$ without changing the magnetic field. This follows from the fact that $\boldsymbol{\nabla} \times (\boldsymbol{A} + \boldsymbol{\nabla}\chi) = \boldsymbol{\nabla} \times \boldsymbol{A}$ because $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = 0$. We use this freedom to make $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ as follows. If $\boldsymbol{\nabla} \cdot \boldsymbol{A} = f(x, y, z) \neq 0$ we define a new vector potential

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi \tag{35}$$

We find that $\boldsymbol{\nabla} \cdot \boldsymbol{A}' = \boldsymbol{\nabla A} + \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\chi = f(x, y, z) + \boldsymbol{\nabla}^2\chi$ where $\boldsymbol{\nabla}^2$ is given by (28). If we require that the scalar function χ is chosen to be the solution of the differential equation $\boldsymbol{\nabla}^2\chi = -f(x, y, z)$ then it follows that $\boldsymbol{\nabla} \cdot \boldsymbol{A}' = 0$. In practice we do not solve the equation for χ. It suffices that we could in principle solve it and thus have $\boldsymbol{\nabla} \cdot \boldsymbol{A}' = 0$. We leave out the prime on $\boldsymbol{A}'$ from now on and we assume henceforwarth that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. The transformation from $\boldsymbol{A}$ to $\boldsymbol{A}'$ in (35) is called a Gauge Transformation and we say that $\boldsymbol{A}$ has been gauged so that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

To obtain an equation for the vector potential $\boldsymbol{A}$ in the situation that the electric field $\boldsymbol{E}$ is either absent or, if it exists, is constant in time we substitute (31) in (20) and get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \mu \boldsymbol{j} \tag{36}$$

We again use the identity (33). In this equation the right hand side may also be written as $\boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})$. In (36) the nabla operators work to the their right on what follows. So we must keep them in the position relative to $\boldsymbol{A}$ where they are in (36). Thus we obtain

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \tag{37}$$

Refer to (28) for the meaning of $\boldsymbol{\nabla}^2$. But $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ so using this in (37) and substituting (37) in (36) we get

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu \boldsymbol{j} \tag{38}$$

This equation looks like (27) except that in (27) both sides are scalars while in (38) both sides are vectors. However, (38) can be written as three equations, one for each component of the vectors on the left and right side. For example, the x component of (38) can be written as

$$\boldsymbol{\nabla}^2 A_x = -\mu j_x \tag{39}$$

This equation is the same as (27) if we replace V by A_x, ρ by j_x and $1/\epsilon$ by μ. We make these substitutions in the solution (30) of (27) and get

$$A_x(\boldsymbol{r}) = \frac{\mu}{4\pi} \int \frac{j_x(\boldsymbol{r}')\,dV}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{40}$$

as a solution of (39). This can be repeated for the y and z components of (38). The resulting two equations and (40) can be combined using a vector notation as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r}')\,dV}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{41}$$

Thus we have found the (particular) solution of (38).

7 Law of Biot and Savart

In the same way we could have derived Coulomb's Law of electrostatics from Gauss' Law (4) in integral from or (17) in differential form we can derive the law of Biot and Savart of magnetostatics from (7) in integral form or (20) in differential form. The term in (7) and (20) involving the derivative of $\boldsymbol{E}$ with respect to time is zero because we consider magnetostatics. Without that term (7) and (20) are called Ampère's Law.

We consider an infinitely small element of electric current $\boldsymbol{j}\,dV = j\,dA\,d\boldsymbol{\ell}$ at position $\boldsymbol{r}'$ and calculate its contribution $d\boldsymbol{A}$ to the vector potential at position $\boldsymbol{r}$. We assume that $\boldsymbol{j}$ and $\boldsymbol{\ell}$ are parallel to each other so

$$j\,d\boldsymbol{\ell} = \boldsymbol{j}\,d\ell \tag{42}$$

Note that we use the same letter $\boldsymbol{A}$ for area and for vector potential. Using (41) we find (no need to integrate over an infinitely small volume element)

$$d\boldsymbol{A} = \frac{\mu}{4\pi}\frac{j(\boldsymbol{r}')\,dA\,d\boldsymbol{\ell}}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{43}$$

We use (31) to calculate the contribution of $d\boldsymbol{A}$ to the magnetic field

$$d\boldsymbol{B} = \boldsymbol{\nabla}\times\left[\frac{\mu}{4\pi}\frac{j(\boldsymbol{r}')\,dA\,d\boldsymbol{\ell}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] = \frac{\mu}{4\pi}j(\boldsymbol{r}')\,dA\,\boldsymbol{\nabla}\times\left[\frac{d\boldsymbol{\ell}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] \tag{44}$$

where we have pulled constants in front of the $\boldsymbol{\nabla}$ operator. The $\boldsymbol{\nabla}$ operator works on $\boldsymbol{r}$, not $\boldsymbol{r}'$. The latter is constant when the $\boldsymbol{\nabla}$ operator does its work on $\boldsymbol{r}$.

We know from vector calculus that

$$\boldsymbol{\nabla}\times(f\boldsymbol{v}) = (\boldsymbol{\nabla}f)\times\boldsymbol{v} + f\boldsymbol{\nabla}\times\boldsymbol{v} \tag{45}$$

Please recall that when guessing this relation it is important to start with $\boldsymbol{\nabla}\times(f\boldsymbol{v})$ and not $\boldsymbol{\nabla}\times(\boldsymbol{v}f)$. We set $f = 1/|\boldsymbol{r}-\boldsymbol{r}'|$ and $\boldsymbol{v} = d\boldsymbol{\ell}$ and use that

$$\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{46}$$

and $\boldsymbol{\nabla}\times d\boldsymbol{\ell} = 0$ because $d\boldsymbol{\ell}$ is constant. With these results (45) becomes

$$\boldsymbol{\nabla}\left(\frac{d\boldsymbol{\ell}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{(\boldsymbol{r}-\boldsymbol{r}')\times d\boldsymbol{\ell}}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \frac{d\boldsymbol{\ell}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{47}$$

where the minus sign is canceled by reversing the order of $(\boldsymbol{r}-\boldsymbol{r}')$ and $\boldsymbol{\ell}$ in their curl. Substitution of (48) into (44) we get

$$d\boldsymbol{B} = \frac{\mu}{4\pi}\, j(\boldsymbol{r}')\, dA \frac{d\boldsymbol{\ell} \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{48}$$

Of course $j(\boldsymbol{r}')\, dA$ can be written as $i(\boldsymbol{r}')$ so alternatively we can write

$$d\boldsymbol{B} = \frac{\mu}{4\pi}\, i(\boldsymbol{r}')\, \frac{d\boldsymbol{\ell} \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{49}$$

Because of (42) we can replace $i\, d\boldsymbol{\ell}$ by $\boldsymbol{i}\, d\ell$ to get

$$d\boldsymbol{B} = \frac{\mu}{4\pi}\, d\ell\, \frac{\boldsymbol{i} \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{50}$$

Equations (48), (49), and (50) are different formulations of the law of Biot and Savart. It is seen that the magnetic field depends upon the distance $|\boldsymbol{r}-\boldsymbol{r}'|$ as $1/|\boldsymbol{r}-\boldsymbol{r}'|^2$ just like the electric field of a charge. The direction of the magnetic field is perpendicular to $d\boldsymbol{\ell}$ and $(\boldsymbol{r}-\boldsymbol{r}')$ and is given by the right-hand rule applied to the curl $d\boldsymbol{\ell} \times (\boldsymbol{r}-\boldsymbol{r}')$. See your favorite textbook for a discussion.

We note that Giancoli in Chapter 28 uses $\boldsymbol{r}$ instead of $\boldsymbol{r}-\boldsymbol{r}'$ and r instead of $|\boldsymbol{r}-\boldsymbol{r}'|$ and introduces the unit vector $\hat{\boldsymbol{r}}$ defined in his notation as

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} \tag{51}$$

The Biot-Savart Law (49) is in his notation

$$d\boldsymbol{B} = \frac{\mu}{4\pi}\, i\, \frac{d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}}{r^2} \tag{52}$$

The unit vector $\hat{\boldsymbol{r}}$ is directed from $\boldsymbol{r}'$ to $\boldsymbol{r}$ as is obvious from our notation.

8 Homework

PROBLEM 1

Explain why a necessary condition for the definition of the potential energy U is that

$$\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0 \tag{53}$$

If (53) is true we call $\boldsymbol{F}$ a conservative force. Make a comparison with the condition

$$\oint dS = \oint \frac{dQ}{T} = 0 \tag{54}$$

required for the entropy S to be defined.

PROBLEM 2

Use Stokes' Theorem the translate the condition (53) into another one: $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$.

PROBLEM 3

Calculate $\boldsymbol{\nabla} \times \boldsymbol{r}$. Is the force $\boldsymbol{F} = \text{constant} \times \boldsymbol{r}$ a conservative force?

PROBLEM 4

Calculate $\boldsymbol{\nabla} \times (\boldsymbol{r}/r^n)$. For which values of n is the force $\boldsymbol{F} = \text{constant} \times (\boldsymbol{r}/r^n)$ a conservative force? Make a comparison with Newton's Law of Gravity and Coulomb's Law of electric forces.

PROBLEM 5

Calculate $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$ by first calculating $\boldsymbol{\nabla} \times \boldsymbol{B}$ and then taking the Divergence of the result.

PROBLEM 6

Show explicitly that

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial \boldsymbol{E}}{\partial t} \right) = \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \tag{55}$$

PROBLEM 7

In Maxwell's Equation (20) take the Divergence on both sides. Use the result of Problem 5 and (55).

PROBLEM 8

Use Maxwell's Equation (17) to eliminate $\boldsymbol{E}$ between it and the result of Problem 7. Show that the resulting equation reads

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \tag{56}$$

PROBLEM 9

Eq. (56) implies Conservation of Charge. To see this, integrate (56) over a closed volume V and differentiate both sides of the resulting equation with respect to t.

a. Show that the first term in (56) can be written as $\partial Q/\partial t$ where Q is the total charge in volume V. Which mathematical property of integration and differentiation did you use?

b. Use Gauss' Law to rewite the second term in (56) as an integral over the surface S that encloses the volume V. What is the physical meaning of this integral?

c. Discuss the physical meaning of (56). What would happen if the term with the Displacement Current were not there? What if the coefficient was anything else but $\mu\epsilon$? Here is another justification for the presence and form of the Displacement Current!