December 10, 2015
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SOLUTIONS TO THE 4C FINAL

PROBLEM 2 (30 points)

Two infinitely long one dimensional charge distributions are positioned along the x-axis and the y-axis. The charge distributions are uniform and their charge densities are specified by $\lambda < 0$. Calculate the electric field everywhere in the x, y plane except on the coordinate axes.

The solution is the sum of two contributions, one from each linear charge distribution. The superposition Principle at work! A single linear charge distribution has an electric field $\boldsymbol{E}$ that is directed radially away from the line of charge with magnitude $E = \lambda/(2\pi\epsilon_0 d)$ where d is the distance from the line of linear charge distribution, see Giancoli, P. 597.
Consider a point P with coordinates $(x, y, 0)$ located in the first quadrant. The point P is a distance y away from the linear charge distribution along the x-axis and a distance x away from the linear charge distribution along the y-axis. The Figure shows the two resulting electric fields, $\boldsymbol{E}_1$ from the charges along the x-axis and $\boldsymbol{E}_2$ from the charges along the y-axis. Their magnitudes are $E_1 = \lambda/(2\pi y)$ and $E_2 = \lambda/(2\pi x)$. To make these two relations hold in all four quadrants we take the absolute values of the magnitudes and generalize the above results to $E_1 = \lambda/(2\pi|y|)$ and $E_2 = \lambda/(2\pi|x|)$. The directions of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ change in an obvious way between the four quadrants. The vector sum of these two is $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and is shown in the Figure. It is left to the reader to calculate its magnitude and specify its direction, for example the angle with the positive x-axis.

PROBLEM 3 (30 points)

An electric dipole is positioned in an inhomogeneous electric field $\boldsymbol{E} = \alpha x \boldsymbol{e}_x$ with $\alpha > 0$ a constant. The dipole has a length ℓ and a dipole moment $\boldsymbol{p}$. The vector $\boldsymbol{p}$ makes an angle β with $\boldsymbol{E}$.

a. Calculate the total force acting on the dipole.

b. Calculate the total torque on the dipole.

c. In view of your answers in a) and b), what is going to happen to the dipole?

This problem differs from the discussion in Giancoli on P. 579-580 in that the electric field is inhomogeneous instead of homogeneous. This causes it to be different at the locations of the two charges that form the dipole. These charges have magnitude $q = |\boldsymbol{p}|/\ell$. The vector $\boldsymbol{p}$ is as always directed from $-q$ to $+q$ and has length ℓ. The Figure shown the geometry of the dipole in the inhomogeneous electric field. The center of the dipole is located at the origin of the coordinate system. The positive charge is located at $x_+ = \ell \cos\beta/2 > 0$ while the negative charge is located at $x_- = -\ell \cos\beta/2 < 0$. The electric fields are $\boldsymbol{E}_+ = \alpha x_+ \boldsymbol{e}_x$ and $E_- = \alpha x_- \boldsymbol{e}_x$. Because of the sign difference between x_+ and x_- the respective electric fields point in opposite directions, different from the case discussed in Giancoli, P. 580. The forces on each of the charges are charge times electric field respectively or $\boldsymbol{F}_+ = +q\boldsymbol{E}_+ = \alpha q \ell \cos\beta/2 \boldsymbol{e}_x$ and $\boldsymbol{F}_- = -q\boldsymbol{E}_- = \alpha q \ell \cos\beta/2 \boldsymbol{e}_x$. $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ are in the same direction and unequal in magnitude! Compare with Giancoli, P. 580 where the two forces are opposite in direction and equal in magnitude.

The two forces do not cancel and there is a net force along $\boldsymbol{e}_x$ that can be calculated from $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ and causes the dipole to move in the direction of the positive x-axis. There is a net torque that can be calculated much like is done in Giancoli, P. 579 for the case of a homogeneous electric field. The rest of the problem is left to the reader.

PROBLEM 4 (20 points)

An electric potential V is given by $V = \alpha/r^2$ where $\boldsymbol{r} = (x, y, z)$ and α is a constant. Calculate the resulting electric field.

$\boldsymbol{E} = -\boldsymbol{\nabla}\cdot V = -\boldsymbol{\nabla}\cdot\alpha/(x^2+y^2+z^2) = [(\partial/\partial x)\boldsymbol{e}_x+(\partial/\partial y)\boldsymbol{e}_y+(\partial/\partial z)\boldsymbol{e}_z]\alpha/(x^2+y^2+z^2)$. The first term involves $(\partial/\partial x)[1/(x^2+y^2+z^2)] = -2x/(x^2+y^2+z^2)^2 = -2x/r^4$. Putting all three terms together and writing $x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z = \boldsymbol{r}$ we get $\boldsymbol{E} = 2\alpha\boldsymbol{r}/r^4$. The magnitude is $E = 2\alpha/r^3$.

PROBLEM 5 (30 points)

An line of charges is positioned along the x-axis between $x = 0$ and $x = L$. The linear charge density $\lambda < 0$ is constant. A point P is located on the x-axis at position $x > L$.

a. Calculate the electric potential at point P.

b. Calculate the electric field at point P.

c. Consider the limit $x >> L$ in your answers of a) and b). Do these two limts make sense? Explain.

d. There is a relation between electric potential and electric field at point P. What is it and do your answers in a) and b) satisy it?

The Figure shows the geometry of the problem. We can not use Coulomb's Law to calculate the potential at P because that law applies only to point charges. We can make it apply if we select an infinitely short piece of length $d\ell$ of the line of charges. It contains an infinitely small charge $\lambda d\ell$ which qualifies as a point charge when we let $d\ell \to 0$. Thus the potential at P from this infinitely small charge is $dV_P = \lambda d\ell/[4\pi\epsilon_0(x-\ell)]$ where ℓ is the location of $d\ell$ along the line of charges and $x-\ell$ is its distance to point P. Note that the denominator is the distance raised to the first power. To get the total potential at P we must integrate over ℓ from 0 to L. This gives $V_P = [\lambda/(4\pi\epsilon_0)]\ln[x/(x-L)]$.
To get the electric field at P we can do one of two things: either differentiate the potential at P with respect to its position x or we can calculate it directly using Coulomb's Law for the electric field of a point charge. We chose the second and defer the first. Following a reasoning similar to the one above we get $d\boldsymbol{E}_P = \lambda d\ell/[4\pi\epsilon_0(x-\ell)^2]$, directed oward the negative x-axis. Note the square in the denominator. The after doing the integral we get $E = [Q/(4\pi\epsilon_0)]/[x(x-L)]$ where $Q = \lambda\ell$. Because $Q < 0$ the electric field is pointing into the negative x-direction.

Taking the limit $x >> L$ (point P is located far away from the line of charges) we find using $1/(1+\epsilon) \approx 1-\epsilon$ and $\ln(1+\epsilon) \approx \epsilon$ with $\epsilon = L/x << 1$ that $V_P = Q/(4\pi\epsilon_0)$ and $E_P = Q/(4\pi\epsilon_0 x^2)$, just as expected. The approximations shown follow from Taylor expansions. You should check them and memorize them. They will be useful for the rest of your professional life.
The relation between electric field and potential is $\boldsymbol{E} = -\boldsymbol{\nabla} V$. It can be verified that this relation holds for the above general results and of course also for the results obtained in the limit $x >> L$.

PROBLEM 6 (30 points)

A dielectric with dielectric constant κ is located between the plates of a planar capacitor. It fills the space between them. With the dielectric in place the capacitance is C. The plates of the capacitor carry charges of $+Q$ and $-Q$ respectively. The capacitor is not connected to anything. An observer pulls out the dielectric. How much work does she have to do pulling it out? Carefully state whether that work is positive or negative.

Let's call the value of the capacitor without the dielectric C_0. The relation between C and C_0 is $C = \kappa C_0$. With the dielectric in place the capacitor's potential energy is $Q^2/(2C)$. Without the dielectric it is $Q^2/(2C_0)$. Note that the charges on the plates do not change when the dielectric is pulled out. The difference in potential energy before and after the dielectric is pulled out is $Q^2/(2C_0)Q^2/(2C) = Q^2/(2C_0)Q^2/(2\kappa C_0) = [Q^2/(2C_0)][(\kappa - 1)/\kappa]$. Because $\kappa > 1$ this difference is potitive so the capacitor contains more potential energy after the dielectric is removed than before. This energy is delivered to the capacitor by the observer doing work equal to this difference and that wotk is thus positive. The observer has to pull.

PROBLEM 7 (20 points)

A beam of electrons of velocity $\boldsymbol{v}$ directed along the $+x$-axis enters a region with a uniform magnetic field in the $+z$-direction. This causes the electron's trajectory to deviate from the direction of the x-axis.

a. What kind of trajectory do the electrons follow when they are in the region of the magnetic field? Specify fully their trajectory, velocity, direction, etc.

b. An observer decides to correct the deviation of the trajectory by applying an electric field in addition to the magnetic field. In which direction should the electric field be and how large should it be so that the electrons continue in a straight line when they enter the region of the magnetic field?

The motion of a charged particle in a magnetic field is discussed in Giancoli, P. 714 - 715. The Lorentz force is $bF = e\boldsymbol{v} \times \boldsymbol{B}$. The velocity's magnitude does not change at all because the force is always perpendicular to the velocity. When the electron enters the region of the magnetic field the cross product points initially in the negative y-direction but the electron's charge is negative making the force point in the positive y-direction. Thus the electron's motion is in the $x - y$ plane as shown in the Figure. The force's magnitude is constant in the region of the magnetic field because the velocity is always perpendicular to the magnetic field so the electron's orbit is part of a circle. After traversing half a circle the electron exits the region of the magnetic field as shown in the Figure and follows a striaght line. The radius of the semicircle is $r = mv/(|e|B)$.

PROBLEM 8 (20 points)

Two parallel conductors (rails) at a distance ℓ have a conducting bar of mass m across it at 90 deg as shown in the figure. The rails are connected to a DC power supply through a switch as shown. The power supply delivers a DC current I. There is a magnetic field $\boldsymbol{B}$ perpendicular to the rails. When the switch is closed the power supply will provide its current instantaneously. The battery (power supply) will maintain the current I despite the induced voltage caused by the movement of the bar.

a. Calculate the force on the bar after the switch is closed.

b. What is the velocity of the bar after a time t?

c. How much energy has the power supply delivered at time t?

The force is $F = \ell I \times \boldsymbol{B}$ whose magnitude is $F = \ell IB$ and direction is to the left. The bar will execute a linear motion of constant acceleration. The usual formulas from 4A mechanics can be used to answer the remaining questions.

PROBLEM 9 (30 points)

A magnetic field is associated with a vector poential $\boldsymbol{A}(x, y, z)$ given by $\boldsymbol{A} = i\, xy\, \boldsymbol{e}_z$ where x and y are spatial coordinates and i is a constant.

a. Calculate the magnetic field.

b. Calculate the magnitude of the magnetic field and specify its direction.

c. Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{A}$.

d. We have defined Gauge Transformations in order to be able to make $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Why did we want that condition to be true?

Using $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ we find $\boldsymbol{B} = i(x\boldsymbol{e}_x - y\boldsymbol{e}_y)$. The magnitude of $\boldsymbol{B}$ is $B = i\sqrt{x^2+y^2}$ so B has the same value along any circle centered on $x = y = 0$. We find $\boldsymbol{e}_x \cdot \boldsymbol{B} = ix$, $\boldsymbol{e}_y \cdot \boldsymbol{B} = -iy$. From these relations the cosine of the angle betwen $\boldsymbol{B}$ and the x-axis and $\boldsymbol{B}$ and the y-axis respectively can be determined. We also find that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{\nabla} \cdot (ixy\boldsymbol{e}_z) = i(\partial/\partial z)(xy) = 0$. This is good because no Gauge Transformation is needed to make it so. The divergence condition is a requirement for the relation between current density and vector potential, see Eq. (41 of)Section 6 of the document "GaussStokes.pdf" in the 4C website.

PROBLEM 10 (20 points)

People who want to measure a magnetic field sometimes use the flip-coil method. They use a coil consisting of N turns, each of area A. Initially the plane of the turns is perpendicular the the unknown magnetic field. They then rotate the coil over 180 deg with angular veocity ω and measure the induced voltage in the coil.

a. What is the voltage as function of time?

b. Calculate the magnetic field from the measured voltage.

Call the angle of the normal to the plane of the coil α. Initially $\alpha = 0$ and the magnetic flux $\phi_B = NAB$ where B is the unknown magnetic field. The magnetic flux as function of time during the 180 deg turn is $\phi_B(t) = NAB\cos(\omega t)$. We take the cosine instead of the sine because we want the correct flux at $t = 0$. The induced voltage in the coil is $\mathcal{E}\Updownarrow\{ = (\partial/\partial\sqcup)\phi_{\lfloor}$. Do not multiply this by N, we have already taken the number of turns into account in calculating the flux. We get $\mathcal{E}\Updownarrow\{ = \mathcal{NAB}\omega\sin\omega\sqcup$ and when the $\mathcal{E}\Updownarrow\{$ is measured we can calculate B. We have been sloppy with the signs. We use Lenz' Law to figure out in which direction the induced voltage sends the current: Such that its effect will counter its cause. Do you agree that its direction depends upon the directin of the turn?

PROBLEM 11 (20 points)

Two coils with self inductances L_1 and L_2 respectively are connected in parallel as shown in the figure.

Calculate the equivalent self inductance of the single coil that replaces the pair.

The solution is similar to the calculation of the equivalent resistor replacing two resistors in parallel. While in that case we dealt with currents, here we must deal with time derivatives of currents because by the definition of L, $\mathcal{E} = \mathcal{L}\text{ị}/\lceil\sqcup$ ignoring a minus sign. We assign three currents in the circuit as shown and use Kirchhoff's Junction Theorem to get $i = i_1 + i_2$. Differentiating this with respect to time we get $di/dt = di_1/dt + di_2/dt$. By definitionwe have $\mathcal{E}_\infty = \mathcal{L}_\infty\lceil\rangle_\infty/\lceil\sqcup$ and $\mathcal{E}_\in = \mathcal{L}_\in\lceil\rangle_\in/\lceil\sqcup$. For the equivalent replacement self inductance L we have $\mathcal{E} = \mathcal{L}\lceil\rangle/\lceil\sqcup$. Substitution into the equation for di/dt and realizing that all three $\mathcal{E}\Updownarrow\{$ are equal (they are between the same two terminals) we can cancel them and get $1/L = 1/L_1 + 1/L_2$.

PROBLEM 12 (30 points).

An inductance, a capacitance, and a switch are connected in a circuit as shown in the Figure. The inductance is L and the capacitance is C. Initially the capacitor is charged and has charges $\pm Q_0$ on its plates. At $t = 0$ the switch is closed.

a. Calculate the differential equation that governs the circuit. Let the dependent variable be the current in the circuit $i(t)$.

b. Solve the differential equation for $i(t)$.

c. Calculate $Q(t)$ from your answer in b).

d. Plot the current in the circuit and the charge on the capacitor as function of time.

This circuit is discussed in Giancoli, P. 793 but Giancoli derives a differential equation with the charge $Q(t)$ on the capacitor as dependent variable. Here we want the current $i(t)$ as the dependent vaiable. There are no junctions so we only need to condider Kirchhoff's Loop Theorem. A positive direction for the current and its derivative as well as a positive direction for going around the loop ar shown in the Figure. We also selected which of the two plates of the capacitor is initially positively charged with charge $Q_0 = Q(0)$. With the switch closed we get $V_c - V_L = Q/C - Ldi/dt = 0$. We see from the signs in the Figure that $i = -dQ/dt$. Differentiation of the differential equation with respect to t and substituting the last relation for i gives $Ld^2i/dt^2 + i/C = 0$. We had this equation in class. It is similar to Giancoli's equation which has the charge Q as dependent variable and it is also the same as the differential equation for a pendulum. It is linear, homogeneous, and second order, thus it has two independent solutions of which we make a linear superposition with two arbitrary constants. You can try exponentials or trogonometric functions of time. To fix the two arbitrary constants we need two initial condtions. One is that an infinitely small time after clsoing the swith the current is still zero so $i(0) = 0$. For the other initial condition we determine di/dt at $t = 0$. From the first equation above (before we diffrentiated it) we see that $di/dt = Q/(LC)$, valid at all t. At $t = 0$ this relation becomes $di/dt = Q_0/(LC)$. In the case of a pendulum these initial conditions correspond to specifying the position of the pendulum and its velocity at $t = 0$. We chose exponentials and try to solve the differential equation by $i(t) = \exp(\gamma t)$. Substitution gives $\gamma^2 = -1/(LC)$ so γ is purely imaginary: $\gamma = \pm j\sqrt{1/(LC)}$ with $j = \sqrt{-1}$. We do not use $i = \sqrt{-1}$ because the letter i is already used for current. Set $\omega^2 = 1/(LC)$ because we will see later that ω corresponds to an angular frequency so the symbol is chosen appropriately. So we have $i(t) = A\exp(+j\omega t) + B\exp(-j\omega t)$ and (differentiating) $di/dt = Aj\omega\exp(+j\omega t) - Bj\omega\exp(-j\omega t)$. Using the two initial conditions at $t = 0$ we find that $A + B = 0$ and $j\omega A - j\omega B = Q_0/(LC) = \omega^2 Q_0$. These two equations can be solved for the two unknowns A and B to get $A = \omega Q_0/(2j)$ and $B = -\omega_0/(2j)$. The current becomes $i(t) = \omega Q_0/(2j)(\exp(j\omega t) - \exp(-j\omega t))$ or $i(t) = \omega Q_0 \sin(\omega t)$. The charge $Q(t)$ on the capacitor is $Q(t) = -\int i(t)dt = Q_0\cos(\omega t) + \text{constant}$, see the earlier discussion for the minus sign. The intial condition on $Q(t)$ is that $Q(0) = Q_0$ so the constant $= 0$ and $Q(t) = Q_0\cos(\omega t)$. This shows that the current and the charge ar $\pi/2$ out of phase. Giancoli has a nice graph showing that on P. 793.

EXTRA CREDIT (30 points)

a. Given a current density $\boldsymbol{j}(\boldsymbol{r})$ write the expression for the resulting vector potential $\boldsymbol{A}(\boldsymbol{r})$.

b. Calculate $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ using $\boldsymbol{A}$ from a).

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\int\frac{\boldsymbol{j}(\boldsymbol{r}')\,d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Next we calculate $\boldsymbol{\nabla} \cdot \boldsymbol{A}$. Note that the $\boldsymbol{\nabla}$ operator acts upon $\boldsymbol{r}$ and not $\boldsymbol{r}'$.

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \int \boldsymbol{j}(\boldsymbol{r}') \cdot \boldsymbol{\nabla} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d^3\boldsymbol{r}' = -\frac{\mu}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}') \, d^3\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

where we used that

$$\boldsymbol{\nabla} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

For the last two equations refer to the document “nablaoperator.pdf” in the 4C website, Eq. (23) setting $f =$ constant because it depends upon $\boldsymbol{r}'$ and Eq. (37). It is not necessary to work the integral because it is seen that the integrand is odd in the integration variable r'.