This document is stored in Documents/4C/vectoralgebra.tex Compile it with LaTex.

September 23, 2014
Hans P. Paar

VECTOR ALGEBRA

1 Introduction

Vector algebra is necessary in order to learn vector calculus. We are dealing with vectors in three-dimensional space so they have three components. The number of spatial variables that functions and vector components can depend on is therefore also three.

I assume that the reader is familiar with vector addition and subtraction as well as multiplication of a vector by a scalar from previous courses.

In this document we review the dot product or scalar product of two vectors and the cross product or vector product of two vectors. These are also familiar from previous courses but I want to introduce the notion of a determinant borrowed from linear algebra, a branch of mathematics that is not a prerequisite for understanding this document. I am unnecessarily fancy in places but you might as well learn about the material in the manner presented below at this time as you have to know it anyway sometime in the future.

We shall use Carthesian coordinate systems which by definition have three mutually perpendicular coordinate axes x, y, z with unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ along its respective x, y, z axes directed in the positive directions. We do not use the notation $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ for these.

2 The Scalar Product of Two Vectors

The scalar product of two vectors $\boldsymbol{v} = (v_x, v_y, v_z)$ and $\boldsymbol{w} = (w_x, w_y, w_z)$ is written formally

$$\boldsymbol{v} \cdot \boldsymbol{w} = \big(v_x \boldsymbol{e}_x + v_y b e_y + v_z \boldsymbol{e}_z\big) \cdot (w_x \boldsymbol{e}_x + w_y b e_y + w_z \boldsymbol{e}_z) \tag{1}$$

We want to work out the parentheses using the distributive law of multiplication. We get nine terms, each of which involves the scalar product of two unit vectors. We now *define* the scalar product of two unit vectors to be a scalar given by

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta which is a scalar that is equal to 1 for $i = j$ and is equal to 0 otherwise. Thus the scalar product of a unit vector with itself is a scalar of magnitude 1 and the scalar product of two different unit vectors is a scalar of magnitude 0.

Working out the parentheses in (1) and using (2) we find that

$$\boldsymbol{v} \cdot \boldsymbol{w} = \big(v_x w_x + v_y w_y + v_z w_z\big) \tag{3}$$

It is seen that the scalar product of two *arbitrary* vectors is also a scalar. The length-squared of a vector $\boldsymbol{v}$ is given by $\boldsymbol{v} \cdot \boldsymbol{v}$ as can be seen from (3). Thus (2) shows that the unit vectors $\boldsymbol{e}_i$ are indeed vectors of unit length. We could have started with (3) as the definition of a scalar product of two arbitrary vectors and work our way back to (2) using that $\boldsymbol{e}_x = (1, 0, 0)$ and so on.

The scalar product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is sometimes defined as

$$\boldsymbol{v} \cdot \boldsymbol{w} = vw \cos\alpha \tag{4}$$

where v and w are the magnitudes of $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively and α the acute angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. To show that the definition (4) is equivalent with everything that preceeds it in this section we start with (3) and choose a coordinate system with its x-axis along the vector $\boldsymbol{v}$ and the y-axis such that the x, y plane contains the vector $\boldsymbol{w}$. Reflect on why we are allowed to do that without a loss of generality. Thus $\boldsymbol{v} = (v_x, 0, 0)$ and $\boldsymbol{w} = (w_x, w_y, 0)$. Using (3) we get $\boldsymbol{v} \cdot \boldsymbol{w} = v_x w_x$. But $v_x = v$ and $w_x = w \cos\alpha$. Eliminating v_x and w_x using the last two equations we see that (3) and (4) are equivalent.

You already knew everything in this section from previous courses. It was merely an introduction to the next section on the vector product of two vectors which contains some new information using determinants that are usually first introduced in a course in linear algebra.

3 The Vector Product of Two Vectors

The vector product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is written formally as

$$\boldsymbol{v} \times \boldsymbol{w} = \big(v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z\big) \times \big(w_x \boldsymbol{e}_x + w_y \boldsymbol{e}_y + w_z \boldsymbol{e}_z\big) \tag{5}$$

We want to work out the parentheses using the distributive law of multiplication. We get nine terms, each of which involves the vector product of two unit vectors. We now *define* the vector product of of two unit vectors to be a vector given by

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \boldsymbol{e}_k \tag{6}$$

where ϵ_{ijk} is the Levi-Civita symbol that is totally antisymmetric in the indices ijk with $\epsilon_{123} = 1$. Antisymmetric in the indices means that if two neighboring indices are exchanged one obtains a minus sign. Thus $\epsilon_{213} = -1$ (one exchange to get 213→123) and $\epsilon_{312} = +1$ (two exchanges to get 312→123). If two or more indices are equal the quantity is zero. This can be seen by exchanging indices until two equal ones are next to each other. Once two equal indices are next to each other their exchange will introduce a minus sign but the quantity will not have changed. It is seen that if one cyclically permutes the three indices that the value of ϵ_{ijk} does not change. In cyclic permutation one rotates the indices as if they are on a circle so $ijk \to jki \to kij \to ijk$. So for example $\boldsymbol{e}_x \times \boldsymbol{e}_x = 0$, $\boldsymbol{e}_x \times \boldsymbol{e}_y = \boldsymbol{e}_z$, $\boldsymbol{e}_x \times \boldsymbol{e}_z = -\boldsymbol{e}_y$, ...

Working out the parentheses in (5) and using (6) we find that

$$\boldsymbol{v} \times \boldsymbol{w} = (v_y w_z - v_z w_y)\boldsymbol{e}_x - (v_x w_z - v_z w_x)\boldsymbol{e}_y + (v_x w_y - v_y w_x)\boldsymbol{e}_z \tag{7}$$

where we grouped terms according to $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$. It is seen that the vector product of two *arbitrary* vectors is a vector.

The magnitude of the vector product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is sometimes defined as

$$|\boldsymbol{v} \times \boldsymbol{w}| = vw \sin\alpha \tag{8}$$

where α is the acute angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ and its direction is perpendicular to the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$. Of the two possible directions one chooses the one obtained with the right-hand rule rotating the first vector $\boldsymbol{v}$ over the acute angle α toward the second vector $\boldsymbol{w}$. To show that this definition is equivalent to everything that preceeds it in this setion we start with (7) and choose the same coordinate system relative to $\boldsymbol{v}$ and $\boldsymbol{w}$ as in Sec. 2 below (4). Using (7) we get $\boldsymbol{v} \times \boldsymbol{w} = v_x w_y \boldsymbol{e}_z$. But $v_x = v$ and $w_y = w \sin\alpha$. Eliminating v_x and w_y using the last two equations we see that (7) and (8) give the same result for $|\boldsymbol{v} \times \boldsymbol{w}|$ and that the direction agrees as well.

Equation (7) is not easy to remember. It can also be written as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \tag{9}$$

The form of the determinant is easier to remember. Put the unit vectors in the top row and the components of the first vector in the product in the second row and he components of the second vector in the third row. The order of the first and second vector matters as will be clear from the discussion but also from a property of determinants.

4 Determinants

An $n \times n$ determinant is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & . \\ a_{21} & a_{22} & . & . \\ a_{31} & a_{32} & . & . \end{vmatrix} \tag{10}$$

The a_{ij} are called elements and $i = [1, n]$ and $j = [1, n]$. The first index labels the row and the second index labels the column in which an element resides. We will only need 2×2 and 3×3 determinants.

The value of a 2×2 determinant is defined as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{11}$$

We will not need this but this equation can also be written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i,j=1}^{2} \epsilon_{ij} a_{1i} a_{2j} \tag{12}$$

This relation is useful for the study of properties of determinants.

The value of a 3×3 determinant is defined the sum of three 2×2 determinants with prefactors as shown

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{13}$$

Note the minus sign and that the prefactors a_{1j} all come from the first row. The 2×2 determinant multiplying each prefactor a_{1j} are obtained by eliminating the first row and the j-th column from the 3×3 determinant. Evaluating the 2×2 determinants and substituting them (13) we obtain

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{14}$$

Apply these rules to (9) and verify that you obtain (7).

We will not need this but (14) can also be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{15}$$

Note that if $\boldsymbol{v}$ and $\boldsymbol{w}$ are exchanged in (9) we get a minus sign. Apparently the determinant in (9) changes sign if row 2 and row 3 are exchanged. You can check this using (15). You will learn in Linear Algebra several other properties of determinants that will be proven using (15). We shall not need these other properties here. The generalisation from 3×3 determinants to $n \times n$ determinants is straightforward using (15). We will not need such determinants here.

As a sanity check we calculate $\boldsymbol{e}_x \times \boldsymbol{e}_z$ using (9)

$$\boldsymbol{e}_x \times \boldsymbol{e}_z = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\boldsymbol{e}_y \tag{16}$$

as expected. It agrees with the alternative definition (8) and below it in magnitude and direction.

5 Applications

We will need an expression for $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$. Using (1) and (9) we get (write out the missing steps)

$$(u_x\boldsymbol{e}_x + u_y\boldsymbol{e}_y + u_z\boldsymbol{e}_z) \cdot \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = u_x \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} + \cdots \tag{17}$$

But the right-hand side equals

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \tag{18}$$

So we have

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \tag{19}$$

We will also need an expression for $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$. There is an elegant way to work this out which is a little beyond the scope of this document and a straightforward way which we will use. Using (9) we find in an obvious notation

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ u_x & u_y & u_z \\ (\boldsymbol{v} \times \boldsymbol{w})_x & (\boldsymbol{v} \times \boldsymbol{w})_y & (\boldsymbol{v} \times \boldsymbol{w})_z \end{vmatrix} \tag{20}$$

The x-component is $u_y(\boldsymbol{v} \times \boldsymbol{w})_z - u_z(\boldsymbol{v} \times \boldsymbol{w})_y$ or $u_y(v_x w_y - v_y w_x) + u_z(v_x w_z - v_z w_x)$. Watch the signs, there are two minus signs that cancel! We sort the terms proportional to v_x and w_x to obtain $v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z)$. The terms between parentheses look almost like the scalar product of $\boldsymbol{u}$ and $\boldsymbol{w}$ in the first term and $\boldsymbol{u}$ and $\boldsymbol{v}$ in the second term. Both terms are missing $v_x u_x w_x$. We can add these in both terms because they cancel owing to the minus sign between the two terms. Doing this we get $v_x(u_x w_x + u_y w_y + u_z w_z) - w_x(u_x v_x + u_y v_y + u_z v_z) = v_x(\boldsymbol{u} \cdot \boldsymbol{w}) - w_x(\boldsymbol{u} \cdot \boldsymbol{v})$. This calculation can be repeated for the y- and z-components of $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$. Multiplying the x-component by $\boldsymbol{e}_x$, the y-component by $\boldsymbol{e}_y$, and the z-component by $\boldsymbol{e}_z$ and summing the three relations we get

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\, \boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{w} \tag{21}$$