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Hans P. Paar

Some Math for Physics 4A

1 Introduction

Physics 4A has Math 20A as prerequisite and Math 20B as corequisite. There are three topics that are not part of these requisites: partial differentiation, total differential, and determinants. We will need the first two early on in 4A when we discuss potential energy. The third topic is needed when we disucss rotations. We will discuss these three items in this document using the math from the requisites and the textbook (Giancoli, see the 4A Syllabus).

2 Partial Differentiation

Differentiation of a function of a single independent variable is known from the requisites. One can also consider functions of more than one variable and differentiate those. Such differentiation is done with respect to one variable at a time while keeping the other variables constant. This is called partial differentiation. It is signified by curly letters ∂ as follows. Assume first that we have a function of two variables $f(x, y)$. The partial differentiation of $f(x, y)$ with respect to x is written as

$$\frac{\partial f(x, y)}{\partial x} \tag{1}$$

and is obtained by differentiating $f(x, y)$ with respect to x while keeping y constant. Of course we could have chosen to partial differentiate $f(x, y)$ with respect to y. How to do that should be obvious.

An an example we take $f(x,y) = xy^2 + 3x^2y$. Partial differentiation with respect to x gives

$$\frac{\partial f(x,y)}{\partial x} = y^2 + 6xy \tag{2}$$

while partial differentiation with respect to y gives

$$\frac{\partial f(x,y)}{\partial y} = 2xy + 3x^2 \tag{3}$$

The extension of partial differentiation of a function of more then two variables should be obvious at this point. For example the function $f(x,y,z)$ can be partially differentiated with respect to either x, y, or z.

3 Total Differential

One can ask for the change in the value of a function $f(x)$ when its independent variable x is changes by an arbitrary but infinitely small amount Δx. Such a change is $f(x+\Delta x) - f(x)$ in the limit $\Delta x \to 0$. We call it df. We know that

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{4}$$

Leaving out the limit, but remembering to take it later, we find that

$$f(x+\Delta x) - f(x) = \frac{df(x)}{dx}\,\Delta x \tag{5}$$

So we have that the total differential of $f(x)$ is

$$df = f(x+\Delta x) - f(x) = \frac{df(x)}{dx}\,\Delta x \tag{6}$$

in the limit $\Delta x \to 0$. Often the limit $\Delta x \to 0$ is left out in favor of replacing Δx by dx, signifying the limit. Thus we get

$$df = f(x+\Delta x) - f(x) = \frac{df(x)}{dx}\,dx \tag{7}$$

This expression looks curously trivial.

Purists among mathematicians will take exception to the sloppy treatment of the limit. You will see that physisicsts do not care about such fine points as long as they get "the right answer".

We are now ready to consider the total differential of a function of more than one variable. Assume first that we have a function of two variables $f(x, y)$. In analogy with the above we define the total differential as the change in the value of a function $f(x, y)$ when its independent variables x and y are simultaneously changed by arbitrary but infinitely small amounts Δx and Δy. Such a change is $df = f(x + \Delta x, y + \Delta y) - f(x, y)$ in the limit $\Delta x \to 0$ and $\Delta y \to 0$. We may subtract and add the same quantity $f(x, y + \Delta y)$ in the expressison for df and get $df = f(x + \Delta x, y + \Delta y) - f(x, y+\Delta y)+f(x, y+\Delta y)-f(x, y)$. Note the location where the subtracted and added terms are put. Now consider the first two of the four terms. It is seen that the second independent variable did not change between the two terms while the first independent variable was changed by Δx. It is as if we have a function of one independent variable only and not of two independent variables. According to the above discussion of a function of one independent variable we may write

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f}{\partial x}\,\Delta x \tag{8}$$

Similary for the last two terms we may write

$$f(x, y + \Delta y) - f(x, y) = \frac{\partial f}{\partial y}\,\Delta y \tag{9}$$

Putting all this together we find for the total differential of $f(x, y)$

$$df = \frac{\partial f}{\partial x}\,\Delta x + \frac{\partial f}{\partial y}\,\Delta y \tag{10}$$

Again the limits $\Delta x \to 0$ and $\Delta x \to 0$ are often left out in favor of replacing Δx by dx and Δy by dy signifying the limits. Thus we get

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{11}$$

As an example we take again $f(x, y) = xy^2 + 3x^2y$ whose partial differentials we calculated in the previous section, see Eq.(2) and Eq.(3). We find that

$$df = (y^2 + 6xy)\,dx + (2xy + 3x^2)\,dy \tag{12}$$

The extension to the total differential of a function of more than two variables should be obvious at this point. We will encounter functions $f(x, y, z)$ of three independent variables x, y, and z. Their total differential is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{13}$$

4 Determinants

Determinants are encountered in many areas of physics, not just in 4A, so you might as well learn about them now and remember them well. An $n \times n$ determinant is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & . \\ a_{21} & a_{22} & . & . \\ a_{31} & a_{32} & . & . \end{vmatrix} \tag{14}$$

The a_{ij} are called elements and $i = [1, n]$ and $j = [1, n]$. The first index labels the row and the second index labels the column in which an element resides. Please note the order of the two indices relative to their meaning. We will only need 2×2 and 3×3 determinants.

The value of a 2×2 determinant is *defined* as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{15}$$

The value of a 3×3 determinant is *defined* as the sum of three 2×2 determinants with prefactors as shown

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{16}$$

Note the minus sign and that the prefactors a_{1j} all come from the first row. The 2×2 determinants multiplying each prefactor a_{1j} are obtained by eliminating the first row and the j-th column from the 3×3 determinant. Evaluating the 2×2 determinants and substituting them we obtain

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{17}$$

The generalisation from 3×3 determinants to $n \times n$ determinants is straightforward but we will not need that in 4A.