4A Final Solutions - Winter 2016

1) $\vec{a} = (5, 2, 0)$, $\vec{b} = (-10\sin 30, 10\cos 30, 0)$
$= (-5, 5\sqrt{3}, 0)$

(a) $\vec{a}+\vec{b} = (5-5, 2+5\sqrt{3}, 0)$
$= (0, 2+5\sqrt{3}, 0)$

(b) $\vec{a}-\vec{b} = (5+5, 2-5\sqrt{3}, 0)$
$= (10, 2-5\sqrt{3}, 0)$

(c) $\vec{b}-\vec{a} = -(\vec{a}-\vec{b}) = (-10, 5\sqrt{3}-2, 0)$

(d) $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$
$= 5(-5) + 2(5\sqrt{3}) + 0(0)$
$= 10\sqrt{3} - 25$

(e) $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

To find θ, use dot product:

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

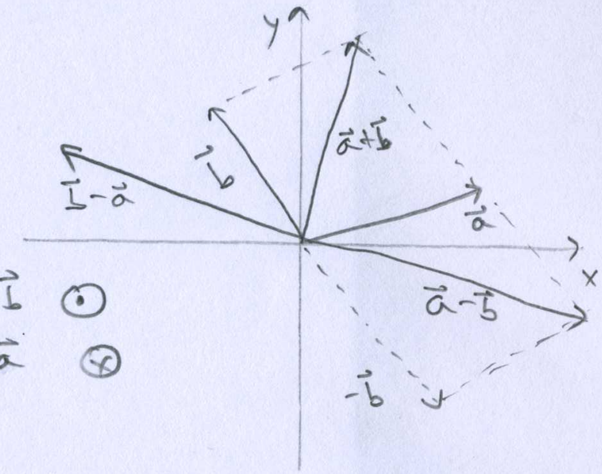
$\Rightarrow \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \dfrac{10\sqrt{3}-25}{\sqrt{29}\,10}$

$\Rightarrow \theta \approx 98.2^\circ$

$\rightarrow \|\vec{a}\times\vec{b}\| = \sqrt{29}\,(10)\sin(98.2^\circ)$
≈ 53.35

points <u>out</u> of page

(f) $\|\vec{b}\times\vec{a}\| = \|\vec{a}\times\vec{b}\| = 53.35$ <u>into</u> page



$\vec{a}\times\vec{b}$ ⊙
$\vec{b}\times\vec{a}$ ⊗

2 $\vec{r} = 3t\hat{i} + (10t - t^2)\hat{j} - (t^3 - 8)\hat{k}$

(a) find t so that $\hat{k}$ component briefly disappears:

$$t^3 - 8 = 0 \Rightarrow t = 2 \text{ seconds}$$

(b) find t so that $\hat{j}$ component disappears:

$$10t - t^2 = 0 \Rightarrow t = 0\,s \;\&\; t = 10\,s$$

(c) $\vec{v} = \dfrac{d\vec{r}}{dt} = 3\hat{i} + (10 - 2t)\hat{j} - (3t^2)\hat{k}$

(d) find t so that $\hat{j}$ component disappears:

$$10 - 2t = 0 \Rightarrow t = 5\,s$$

(e) $\vec{a} = \dfrac{d\vec{v}}{dt} = -2\hat{j} - 6t\,\hat{k}$

(f) $\vec{a}$ is never zero as there is always a $\hat{j}$ component.

3 $1\,lb = 4.45\,N \Rightarrow 200\,lbs = 890\,N$

$1\,mile = 1.609\,km \Rightarrow 3950\,miles = 6357\,km$

$R_E = 6357\,km$, $R_M = 3400\,km$,

(a) $W_E = M_E g_E \Rightarrow M_E = \dfrac{890\,N}{10\,m/s^2} = 89\,kg$

(b) $W_M = M_E g_M = 89\,kg \cdot 3.7\,m/s^2 = 329.3\,N$
$= 74\,lbs$

(c) mass doesn't change:

$$M_M = M_E$$

(d) $g_M = \dfrac{G M_M}{R_M^2}$ & $g_E = \dfrac{G M_E}{R_E^2}$

$$\Rightarrow \frac{M_E}{M_M} = \left(\frac{R_E}{R_M}\right)^2 \frac{g_E}{g_M}$$

$$= \left(\frac{6357}{3400}\right)^2 \left(\frac{10}{3.7}\right)$$

$$= 9.45$$

4 (a) In an inelastic collision, KE is not conserved by definition $\Rightarrow$ total mechanical energy not conserved

(b) $\xrightarrow{m, v}$ $\xleftarrow[m, v]{}$ $\bullet\bullet$

$p_i = 0 \Rightarrow p_f = 0$

$\rightarrow v_f = 0$

* Conservation of momentum used (valid since $F_{ext} = 0$)

(c) $E_i = 2(\tfrac{1}{2})mv^2 = mv^2 \rightarrow E_f = 0$

* not conserved because inelastic collision

(d) Total energy is conserved (as contrasted from total <u>mechanical</u> energy). The lost mechanical energy goes to heat so that total energy still conserved.

(e)(f) Total energy momentum conserved. Initially, momentum vectors are equal & opposite. Because this is elastic collision, final momentum vectors are equal & opposite.

$\|\vec{v}_f\| = \|\vec{v}_i\|$

5 (a) $F_s \downarrow \downarrow F_g$

$g = +9.8\ m/s^2$

$F_s + F_g = ma_c$

$k\Delta x + mg = mv^2/r, \quad \Delta x \equiv r - l$

$\Rightarrow r = \Delta x + l$

So, $k\Delta x^2 + (kl + mg)\Delta x + mgl - mv^2 = 0$

$$\Rightarrow \Delta x = \frac{-(kl+mg) \pm \sqrt{(kl+mg)^2 - 4k(mgl - mv^2)}}{2k}$$

want this to be positive, so take the positive square root.

(b) $+\uparrow$ $\uparrow F_s$ $\downarrow F_g$ $\Rightarrow k\Delta x - mg = mv^2/r$

$k\Delta x^2 + (kl - mg)\Delta x - mgl - mv^2 = 0$

$$\Rightarrow \Delta x = \frac{-(kl-mg) + \sqrt{(kl-mg)^2 + 4k(mgl + mv^2)}}{2k}$$

(c) $+\downarrow$ $\downarrow F_s$ $\quad k\Delta x = mv^2/r$

$\Rightarrow k\Delta x^2 + kl\Delta x - mv^2 = 0$

$$\Delta x = \frac{-kl \pm \sqrt{(kl)^2 + 4kmv^2}}{2k}$$

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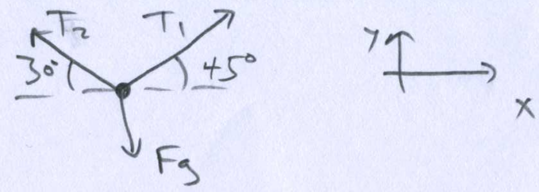


$\hat{x}$: $T_1 \cos 45 - T_2 \cos 30 = ma_x = 0$

$\Rightarrow \frac{T_1}{\sqrt{2}} - \frac{T_2\sqrt{3}}{2} = 0$

$\hat{y}$: $T_1 \sin 45 + T_2 \sin 30 - mg = ma_y = 0$

$\Rightarrow \frac{T_1}{\sqrt{2}} + \frac{T_2}{2} - mg = 0$

$\rightarrow \quad T_2 = \frac{2mg}{1+\sqrt{3}}$ & $T_1 = \frac{\sqrt{6}\, mg}{1+\sqrt{3}}$

(b) Tensions not equal because problem not symmetric.

7 $v(t) = A\cos(\omega t + \phi)$

(a) $x(t) = \int v(t)\,dt$

$= \frac{A}{\omega}\sin(\omega t + \phi) + C$

To find the constant C:

$x(0) = \frac{A}{\omega}\sin(\phi) + C$

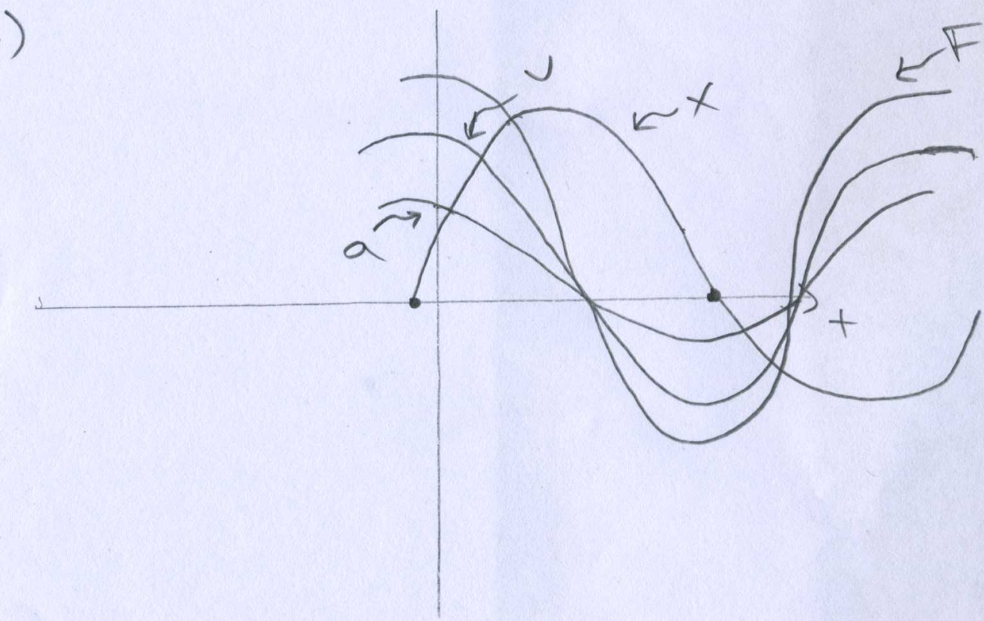
$\Rightarrow C = x(0) - \frac{A}{\omega}\sin(\phi)$

So, $x(t) = \frac{A}{\omega}\left[\sin(\omega t + \phi) - \sin(\phi)\right] + x(0)$

(b) $a(t) = \frac{dv}{dt} = -A\omega\sin(\omega t + \phi)$

(c) $F(t) = ma(t) = -mA\omega\sin(\omega t + \phi)$

(d)



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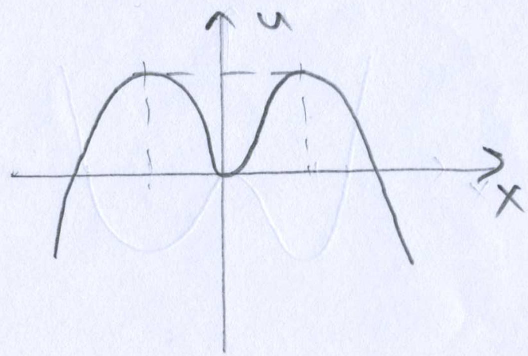
8 $u(x) = Ax^4 + Bx^2$, $A < 0$ & $B > 0$

(a) $\frac{du}{dx} = 0 = 4Ax^3 + 2Bx$

$\Rightarrow x = 0$ & $x = \pm\sqrt{-\frac{1}{2}\frac{B}{A}}$



(b) $\vec{F} = -\frac{du}{dx}\hat{\imath} = -(4Ax^3 + 2Bx)\hat{\imath}$

(c) Force is zero at $x = 0, \pm\sqrt{-\frac{1}{2}\frac{B}{A}}$

(d) $u_{max} = u\left(x = \pm\sqrt{-\frac{1}{2}\frac{B}{A}}\right) = -\frac{B^2}{4A}$

require $\frac{1}{2}mv_0^2 > -\frac{B^2}{4A}$

$\Rightarrow v_0 > \sqrt{-\frac{B^2}{2mA}}$

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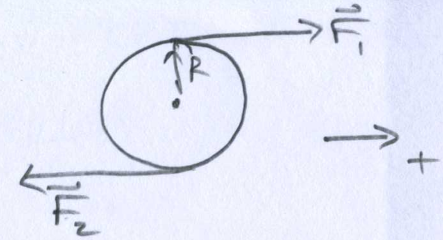

$\|\vec{F}_1\| \neq \|\vec{F}_2\|$

(a) $Ma_{com} = \Sigma F_{ext}$

$\Rightarrow a_{com} = \frac{|F_1| - |F_2|}{M}$

(b) Disk will also rotate as there is a net torque

(c) $\tau_{net} = I\alpha \Rightarrow \alpha = \frac{\tau_{net}}{I}$

$= \frac{R|F_1| + R|F_2|}{\frac{1}{2}MR^2}$

$= \frac{2(|F_1| + |F_2|)}{MR}$

(d) Angular momentum not conserved since τ_{net} external $\neq 0$.

/s

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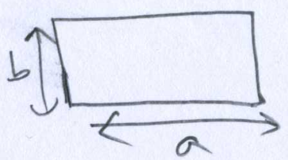
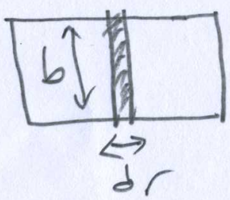


(a) COM is in the middle of the plate. (i.e. for a coord. system in the lower left with right $= +\hat{\imath}$ direction, up $= +\hat{\jmath}$ direction.)

Then, $\bar{r}_{com} = \left(\frac{a}{2}, \frac{b}{2}\right)$.

(b)



$I = \int r^2 \, dm$

$\frac{dm}{dA} = \sigma \Rightarrow dm = \sigma \, dA = \sigma b \, dr$

$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} r^2 \sigma b \, dr = \sigma b \left(\frac{1}{3}\right) r^3 \Big|_{-a/2}^{a/2} = \frac{1}{12} M a^2$

(c) $I = \int_0^a r^2 \sigma b \, dr = \sigma b \left(\frac{1}{3}\right) r^3 \Big|_0^a = \frac{1}{3} M a^2$

* Note, I'm using $\sigma = \frac{M}{ab}$

(d) $I = I_{com} + M R^2$

$= \frac{1}{12} M a^2 + M \left(\frac{a}{2}\right)^2$

$= \frac{1}{3} M a^2$ ✓

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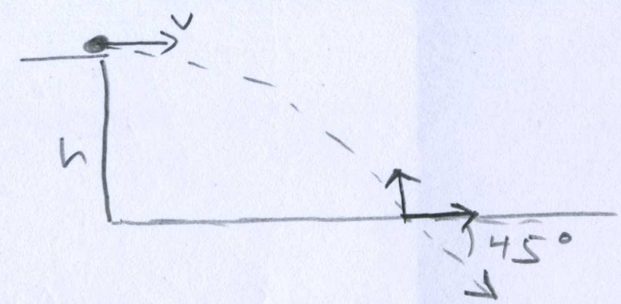

The $\theta = 45°$ means the x & y components of v_f are equal. Since there are no forces in the x-direction $v_{fx} = v$. So,

$v_f = (v, -v)$ in the coord. system drawn

By $E_i = E_f$:

$\frac{1}{2} m v^2 + m g h = \frac{1}{2} m (v_f)^2$

$\Rightarrow h = \frac{1}{2g} v^2$

* $(v_f)^2 = 2 v^2$

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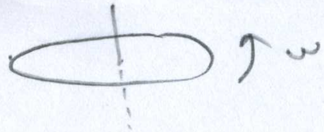

(a) $I_{ring} = \int r^2 dm = mR^2 \rightarrow$ all mass is @ radius R.

(b) $I_{total} = I_{ring} + I_{ball}$
$= MR^2 + mR^2$

(c) $L_i = L_f$ ~ because $\Sigma \tau_{ext} = 0$.

$I_{ring} \omega = I_{total} \omega_f$

$\rightarrow \omega_f = \omega \frac{I_{ring}}{I_{total}} = \omega \frac{MR^2}{R^2(m+M)}$
$= \omega \frac{M}{M+m}$

(d) yes, $L_i = L_f$ because $\Sigma \tau_{ext} = 0$.

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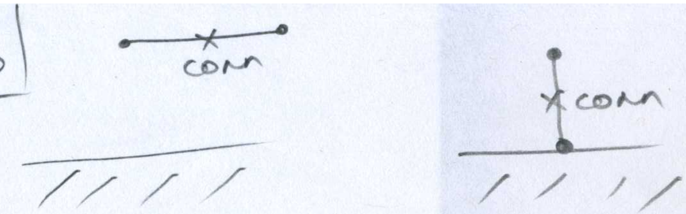


$E_i = mgl$ $\qquad$ $E_f = mg\frac{l}{2} + KE_{rot} + \Delta$

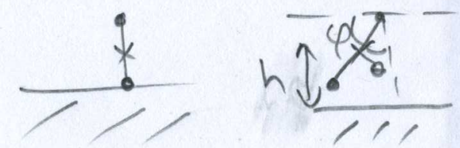
$mgl = mg\frac{l}{2} + KE_{rot} + \Delta$

$\Rightarrow KE_{rot} = \frac{1}{2}mgl - \Delta$

(b) max Δ occurs when $KE_{rot} \rightarrow 0$

$\Rightarrow \frac{1}{2}mgl = \Delta_0$

(c) If $\Delta < \Delta_0$, then bar still has some rotational KE.



$E_i = KE_{rot} + \frac{mgl}{2}$ $\qquad$ $E_f = mg\frac{l}{2}(1-\cos\theta)$
$= mgl - \Delta$ $\qquad$ $= mgh$

$\Rightarrow mgl - \Delta = mgh \Rightarrow h = l - \frac{\Delta}{mg}$

or $mgl - \Delta = mg\frac{l}{2}(1-\cos\theta) + mg\frac{l}{2}$

$\Rightarrow \theta = \cos^{-1}\left[\frac{2\Delta}{lmg}\right]$

$\varphi = \frac{\pi}{2} - \theta$.

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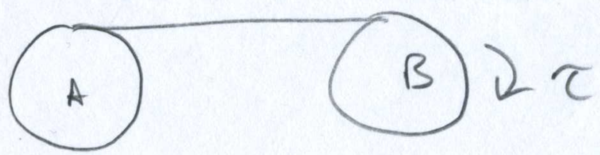

Let's say clockwise = positive rotation

$\Sigma \tau_B = \tau - TR = I\alpha$

$\Sigma \tau_A = TR = I\alpha$

$\Rightarrow \tau - TR = TR$

$\rightarrow |T| = \frac{\tau}{2R}$

(b) $\tau - \frac{\tau}{2} = mR^2 \alpha$

$\Rightarrow \alpha = \frac{\tau}{2mR^2}$

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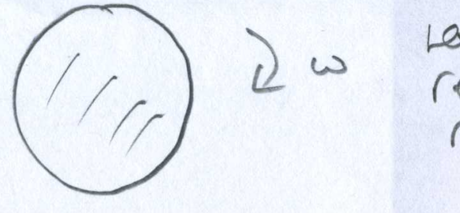

Let's say clockwise represents positive rotation

(a) @ $t=0$, $\omega = 0$

$\tau_{net} = \tau = I\alpha$, $I = \frac{1}{2}mR^2$

$\Rightarrow \alpha = \frac{2\tau}{mR^2}$

(b) @ $t = \infty$, $\alpha = 0$

$\tau_{net} = \tau - \tau_f = 0$

$\Rightarrow \omega = \frac{\tau}{A}$

(d) $\tau - \tau_f = I\alpha$

$\tau - A\omega = \frac{1}{2}mR^2 \frac{d\omega}{dt}$

(e) $\int dt = \int \frac{1}{2}mR^2 \frac{d\omega}{\tau - A\omega}$

$\Rightarrow t = \frac{1}{2}mR^2 \ln(\tau - A\omega)\left(-\frac{1}{A}\right) + C$

$\Rightarrow -C - \frac{2At}{mR^2} = \ln(\tau - A\omega)$

$\Rightarrow \omega = \frac{1}{A}\left(\tau - De^{-2At/mR^2}\right)$

* The constant C is an integration constant. $e^{-C} = D$. D must be τ so that $\omega = \tau/A$ for $t=0$.

$\omega = \frac{\tau}{A} e^{-2At/mR^2}$

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