H.P. Paar **PHYS 4B: Mechanics, Fluids, Waves & Heat** Spring 2016

Quizz # 1 solutions

Solutions by Yury Kiselev

1. (40 points)

   (a) (15 points) The bar is stable if center mass of the construction is above the table. The center of mass position, relative to the left side of the bar is given by $x_{CM} = \frac{Mx_M + mx_m}{M+m}$, where x_M is a center of mass of M, so relative to the left side, $x_M = l/2$ and $x_m = l$. So $x_{CM} = \frac{Ml/2+ml}{M+m}$. The value of x is $x = l - x_{CM} = \frac{Ml/2}{M+m}$.

   (b) (10 points) The edge of the table is at the center of the bar, so $I_{bar} = Ml^2/12$. We can prove that by calculation: $I = \displaystyle\int r^2 \mathrm{d}m = 2\int_0^{l/2} x^2 \mathrm{d}x \cdot (M/l) = Ml^2/12$.

   (c) (15 points) We can write 2nd Newton's law for the case of rotation: $I_{tot}\alpha = \tau$, where I_{tot} — total inertial moment around the edge of the table, α — angular acceleration and τ — total torque. $I_{tot} = I_{bar} + I_m$, calculated relative to the center of the bar, so $I_{tot} = Ml^2/12 + m(l/2)^2$. Torque is only created by gravitational force, acting on mass m, because lines of gravitational force, acting on M and the normal force, both go through the axis of rotation, edge of the table. Then, $\tau = mgl/2$ and angular acceleration is $\alpha = \frac{mgl/2}{Ml^2/12 + m(l/2)^2} = \frac{mgl}{Ml^2/6 + ml^2/2}$.

2. (30 points)

   (a) (10 points) Below the reader can find not a simple solution, but a discussion of paths that can lead us to the correct one.

   In the calculations let's project all vectors to y axis, pointing upwards. We need to be very careful with signs in this problem, though there are different ways to obtain a correct result. We say that the force exerted by the top spring on the mass is $F_1 = +k_1\Delta l_1$, where $\Delta l_1 = y_1 - y$. Here we call y_1 a position of mass m on y axis that corresponds to unstretched first spring. Analogously for the second spring, $F_2 = -k_2\Delta l_2$ because it's directed in the opposite way. $\Delta l_2 = y - y_2$.

   According to 2nd Newton's law, $F_{tot} = ma$, so $m\frac{\partial^2 y}{\partial t^2} = -y(k_1 + k_2) + k_1 y_1 + k_2 y_2$. The constant term $k_1 y_1 + k_2 y_2$ depends on where we choose zero position for the y-axis and will vanish if we choose zero position at the equilibrium. To see that, let's find position of the equilibrium in our setting: $k_1\Delta l_1 = k_2\Delta l_2$ for the equilibrium, which results in $y_{eq} = \frac{k_1 y_1 + k_2 y_2}{k_1 + k_2}$. Now if we make a change of variables: substitute $y = z + y_{eq} = z + \frac{k_1 y_1 + k_2 y_2}{k_1 + k_2}$, the equation of motion will become $m\frac{\partial^2 z}{\partial t^2} = -z(k_1 + k_2)$ or $m\frac{\partial^2 z}{\partial t^2} + z(k_1 + k_2) = 0$.

   (b) (5 points) This is second order linear differential equation. It's homogeneous if we chose such vertical variable that eliminates constant term and non-homogeneous otherwise.

(c) (10 points) We will be able to solve the equation, if we get rid of the constant term. Then, using above, $z = A\cos\omega t + B\sin\omega t$, where A and B are arbitrary constants. Alternative form of the solution is $z = A\cos(\omega t + \phi)$, where A and ϕ are arbitrary constants.

If your definition of the vertical coordinate results in a constant term in DE, as is the case for variable y, we will get $y = y_{eq} + A\cos\omega t + B\sin\omega t = \frac{k_1 y_1 + k_2 y_2}{k_1+k_2} + A\cos\omega t + B\sin\omega t = \frac{k_1 y_1 + k_2 y_2}{k_1+k_2} + A\cos(\omega t + \phi)$. Any of this answers is acceptable.

(d) (5 points) At this point, we can only specify $\omega = \sqrt{\frac{k_1+k_2}{m}}$. All other constants are unknown (y_{eq}, y_1, y_2) or depend on the initial conditions (A, B or ϕ).

3. (30 points)

(a) (10 points) The system satisfies equation $F = ma$, where $F = -kx$ and $a = \frac{\partial^2 x}{\partial t^2}$, so $\frac{\partial^2 x}{\partial t^2} + \frac{k}{m}x = 0$.

(b) (5 points) The solutions of the equation above are $\cos\omega t$ and $\sin\omega t$, where $\omega = \sqrt{\frac{k}{m}}$, so the frequency is $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. [Both f and w are ok for the final answer in part b]

(c) (10 points) One way to find an amplitude of the oscillation is by invoking conservation of energy. After the push, energy of the system is $E = U + K = 0 + \frac{mv_0^2}{2}$. At the maximum stretch point (where $x = A$), energy is $E = U + K = kA^2/2 + 0$. They are equal, because there is no damping (surface is frictionless and we neglect air resistance), and so $kA^2/2 = mv_0^2/2$, or $A = v_0\sqrt{\frac{m}{k}}$. We could also find the same result for the amplitude from considering initial conditions. Coordinate $x(t) = A\cos(\omega t + \phi)$, and a velocity is a time derivative of $x(t)$, then $v(t) = -A\omega\sin(\omega t + \phi)$. The initial conditions are $x(t = 0) = 0$ and $v(t = 0) = -v_0$. The first one gives us $\cos(\phi) = 0$, so $\phi = \pm\pi/2$, and the second is $v(t = 0) = -A\omega\sin(\phi) = -v_0$, so $\phi = \pi/2$ and $A = v_0/\omega = v_0\sqrt{\frac{m}{k}}$. We obtained the same result.

(d) (5 points) No, it is not damped, the surface is frictionless and we usually neglect air resistance.

4. (extra 30 points)
If the train moves with constant velocity, it's still an *inertial* system, so the physical laws are the same as is in a train at rest (Newton's first Law). The frequency didn't change and remains $\omega = \sqrt{\frac{g}{l}}$. The reason why it oscillates is not important – it's either have residual momentum from initial push, or it may oscillate because of small bumps of the rails.