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# LONGITUDINAL WAVES

I discuss the derivation of the wave equation for longitudinal waves. The discussion will be more extensive then in class and in Giancoli.

## 1 Wave Equation

We have discussed transverse waves and derived the transverse wave equation in class. It reads

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

This partial DE can be found in Giancoli Sec. 15-5. This equation has $x$, the location along the propagating medium, and $t$ as independent variables and $y = y(x,t)$, the displacement of the propagating medium, as dependent variable. The wave propagates along the $x$-direction with a velocity $v$ and the displacement $y$ is perpendicular the the direction of propagation. Hence the name "Transverse Wave". The mean value of $y$ is zero because on average the propagating medium does not get displaced but it merely oscillates about its $y = 0$ equilibrium position. We assume that the displacement $y$ and its derivatives are small. The velocity of the transverse displacement $y(x,t)$ is $\partial y/\partial t$ and is not the same as the propagation velocity $v$ of course.

When we consider a longitudinal wave propagating along the $x$-direction we need a coordinate like $y$ we had in the case of a transverse wave that describes the motion of the propagating medium parallel to the propagation direction. We call this coordinate $s$ and it is a function of $x$ and $t$ just like $y(x,t)$ for a transverse wave, so $s = s(x,t)$. As with the transverse wave, the mean value of $s$ is zero because on average the propagating medium does not get displaced but it merely oscillates about its $s = 0$ equilibrium position. We assume throughout that the displacement $s$ and its derivatives are small. Also the velocity of the longitudinal displacement $s(x,t)$ is $\partial s/\partial t$ and is not the same as the propagation velocity $v$ of course.

Longitudinal waves are pressure variations $P_e$ that propagate because the propagating medium wants to recover its equilibrium position and equilibrium pressure $P_0$ when it is displaced by a pressure variation away from the equilibrium pressure. The pressure variation causes a change in mass density $\rho$ of the propagating medium. The pressure $P$ is a function of $\rho$ so we can write

$$P = f(\rho) \tag{2}$$

For the equilibrium values $P_0$ and $\rho_0$, the pressure and density in the absence of a wave, we have

$$P_0 = f(\rho_0) \tag{3}$$

If the pressure changes by a small amount $P_e$ because of the passage of a longitudinal wave that changes the density by $\rho_e$ from their equilibrium values we have according to (2)

$$P_0 + P_e = f(\rho_0 + \rho_e) \tag{4}$$

In general we can make a Taylor Series expansion of $f(x+\epsilon)$ around $f(x)$ as follows

$$f(x+\epsilon) = f(x) + \epsilon \frac{df}{dx}|_x + O(\epsilon^2) \tag{5}$$

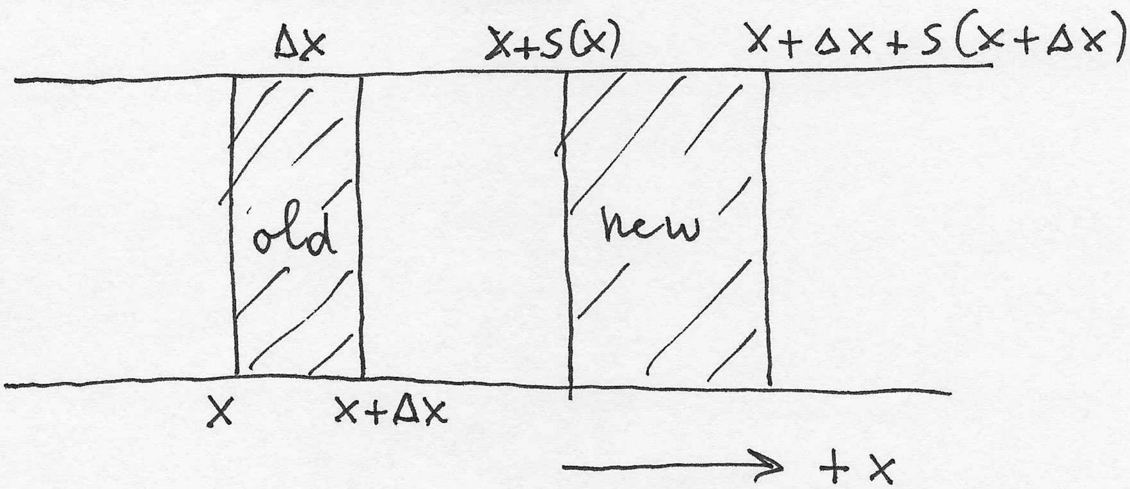
where we truncated the series after the second trm because we assume that $\epsilon$ is small. Using this in (4) we get

$$P_0 + P_e = f(\rho_0 + \rho_e) = f(\rho_0) + \rho_e \frac{dP}{d\rho}|_{\rho_0} \tag{6}$$

Using (3) in (6) we get

$$P_e = \rho_e \frac{\partial P}{\partial \rho}|_{\rho_0} \tag{7}$$

We now consider an infinitely thin slab of the propagating medium at two different times, once before a wave passes through and once when a wave passes through. The slab has area $A$. They are shown much displaced from each other for clarity. In reality they will be overlapping. Initially in the absence of a wave, the slab has a thickness $\Delta x$ with one side located at $x$ and the other side at $x + \Delta x$. The sides of each slab are wave fronts that are straight lines perpendicular to the sides of the propagating medium because the source of the waves is infinitely far away. Initially the slab has a mass $\rho_0 A\, \Delta x$. When a wave passes through the slab, each wave front moves as a whole along the propagation direction.




The wavefront at position $x$ moves to position $x + s(x,t)$ while the wavefront at position $x + \Delta x$ moves to position $x + \Delta x + s(x + \Delta x, t)$. Note the notation and also that the two wave fronts in general will move differently. Their movements are decribed by the variable $s(x,t)$ as defined earlier. The mass of the new slab is $\rho A\, [x + \Delta x + s(x + \Delta x, t) - (x + s(x,t))]$. Because both slabs have the same mass we have

$$\rho_0 A\, \Delta x = \rho A\, \left[x + \Delta x + s(x + \Delta x, t) - (x + s(x,t))\right] \tag{8}$$

Because $\Delta x$ is small we again do a Taylor Series expansion

$$s(x + \Delta x, t) = s(x,t) + \Delta x \frac{\partial s}{\partial x} + O[(\Delta x)^2] \tag{9}$$

where we truncated the series after the second term because we assume that $\Delta x$ is small. Using this result in (8) we get

$$\rho_0 A\, \Delta x = \rho A\, [\Delta x \frac{\partial s}{\partial x} + \Delta x] \tag{10}$$

or simplifying

$$\rho_0 = \rho \frac{\partial s}{\partial x} + \rho \tag{11}$$

Using $\rho = \rho_0 + \rho_e$ we get

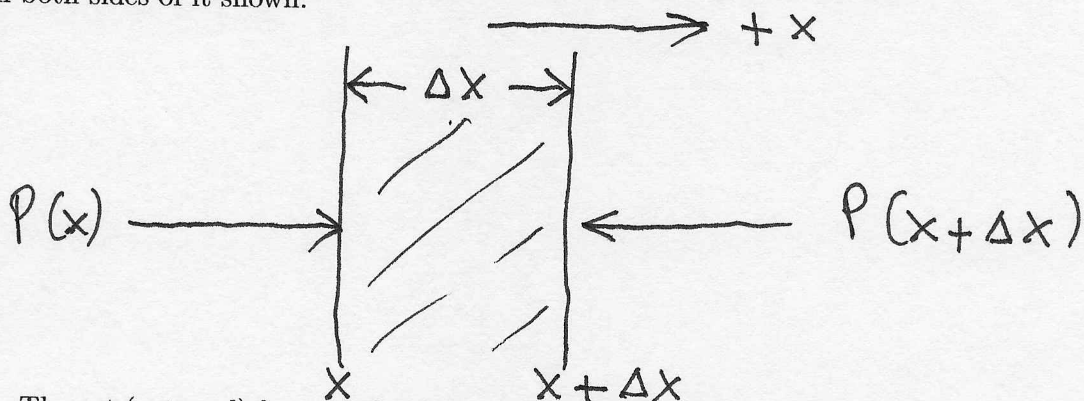
$$\rho_0 = \rho_0 \frac{\partial s}{\partial x} + \rho_e \frac{\partial s}{\partial x} + \rho_0 + \rho_e \tag{12}$$

But both $\rho_e$ and $\partial s/\partial x$ are small so their product is quadratically small so we drop the $\rho_e \partial s/\partial x$ term to get

$$\rho_0 \frac{\partial s}{\partial x} + \rho_e = 0 \tag{13}$$

This equation makes sense: if $\partial s/\partial x$ is positive, the slab got stretched by the passge of the wave and the change in density $\rho_e$ is negative and vice versa.

We are now ready to derive the wave equation for longitudinal waves. As was the case for transverse waves, we use Newton's second law. In the Figure we show a slab with area $A$ of the propagating medium with pressures on both sides of it shown.




The net (external) force on the slab is $P(x)A - P(x+\Delta x)A$ and this force is acting on a slab with mass $\rho_0 A\,\Delta x$. Note that we use an approximation when we use $\rho_0$ instead of $\rho$ but the difference $\rho_e$ between the two can be neglected. Once again we use a Taylor Series expansion to expand $P(x+\Delta x)$

$$P(x+\Delta x) = P(x) + \Delta x \frac{\partial P}{\partial x}|_x + O[(\Delta x)^2] \tag{14}$$

Using this in the equation for the net force, we get that the net force is $P(x)A - P(x+\Delta x)A = P(x)A - [P(x) + \partial P/\partial x\,\Delta x]A = -\partial P/\partial x\,A\,\Delta x = -\partial(P_0 + P_e)\,A\,\Delta x = -\partial P_e/\partial x\,A\,\Delta x$ where we used that $P_0$ is constant independent of $s$. The acceleration of the slab is $\partial^2 s/\partial t^2$. Using the net force, the slab's mass, and the slab's acceleration we get

$$\rho_0 A\,\Delta x \frac{\partial^2 s}{\partial t^2} = -\frac{\partial P_e}{\partial x} A \Delta x \tag{15}$$

or

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial P_e}{\partial x} \tag{16}$$

Using (7) we get

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \rho_e}{\partial x}\frac{\partial P}{\partial \rho}|_{\rho_0} \tag{17}$$

Note that the factor $\partial P/\partial \rho|_{\rho_0}$ is a fixed quantity.

We now use (13) into (17) to get

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = \rho_0 \frac{\partial^2 s}{\partial t^2}\frac{\partial P}{\partial \rho}|_{\rho_0} \tag{18}$$

Canceling $\rho_0$ and rearranging we get

$$\frac{\partial^2 s}{\partial x^2} - \frac{1}{\frac{\partial P}{\partial \rho}|_{\rho_0}}\frac{\partial^2 s}{\partial t^2} = 0 \tag{19}$$

This equation has the standard form of a wave equation that describes longitudinal waves whose propagation is described by the dependent variable $s(x,t)$, compare with (1) for transverse waves where the dependent variable is $y(x,t)$. The velocity of propagation is given by

$$v^2 = \frac{\partial P}{\partial \rho}|_{\rho_0} \tag{20}$$

## 2 Propagation Velocity

To get the propagation velocity of longitudinal waves we consider the case of a gas as propagating medium. We need to calculate $\partial P/\partial \rho$. We will consider an ideal gas described by $PV = NkT$ where $P$ is the pressure, $V$ the volume, $N$ the total number of atoms or molecules in the volume $V$, $k$ the Boltzmann constant, and $T$ the absolute temperature. Newton was the first to calculate the propagation velocity of longitudinal wavs in a gas but he made a wrong assumption. He assumed that the local temperature $T$ of a slab would be constant during the passage of a wave. This is not the case, the temperature changes during such passage and we need to consider adiabatic changes (no heat flow) of the state of the gas in the slab. This was done by Laplace 100 years later.

Let's do Laplace's calculation. It will turn out that with a small change Newton's wrong result can be obtained from Laplace's solution. We need to consider adiabatic changes of the state of a gas. It is described by

$$PV^{\gamma} = \text{constant} \tag{21}$$

Here $\gamma$ is a constant greater then 1 whose value depends upon the nature of the gas. For a monoatomic gas (a nobel gas) $\gamma = 5/3$ and $\gamma$ is smaller for multi-atomic gases. Because the gas' density $\rho$ is inversely proportional to $V$ we an write (21) as

$$P = \text{constant}\,\rho^{\gamma} \tag{22}$$

Diffentiation gives

$$\frac{\partial P}{\partial \rho} = \frac{\gamma P}{\rho} \tag{23}$$

Multiplying numerator and denominator by $V$ and using $PV = NkT$ we get

$$\frac{\gamma P}{\rho} = \frac{\gamma NkT}{\rho V} = \frac{\gamma NkT}{Nm} = \frac{\gamma kT}{m} \tag{24}$$

where we used that $\rho V$ is the total mass of the gas in volume $V$ and that this equals the total number of atoms or molecules $N$ in the volume $V$ times their mass $m$. So the velocity of propagation of longitudinal waves using (20) is

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma kT}{m}} \tag{25}$$

So the propagation velocity depends upon pressure and temperature as well as on the nature of the gas.

We know that the average velocity squared of atoms or molecules in a gas is given by $< u^2 >= 3kT/m$. Here we use $u$ for the velocity of atoms or molecules in a gas to distinguish it from the propagation velocity $v$. Thus $kT = m < u^2 > /3$ and squaring (25) and substitution of $kT$ gives

$$v^2 = \frac{1}{3}\gamma < u^2 > \tag{26}$$

or for a mono-atomic gas $v^2 = (5/9) < u^2 >$. The quantities $v^2$ and $< u^2 >$ are nearly equal and that is no coincidence as it is the oscillatory longitudinal motion of the atoms or molecules in the propagating medium that cause the propagation of the wave.

We can obtain Newton's incorrect result easily as follows. He assumed that the temperature $T$ of the slab does not change as the longitudinal wave passes through the slab. This we have

$$PV = \text{constant} \tag{27}$$

Comparing with (21) we see that if we set $\gamma = 1$, (27) and (21) are identical. Thus we can take the results of (25) and (26) and set $\gamma = 1$. We obtain Newton's results

$$v = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{kT}{m}} \tag{28}$$

and

$$v^2 = \frac{1}{3} < u^2 > \tag{29}$$

Newton was not very wrong! He too obtained the result that the propagation velocity depends upon pressure and temperature as well as on the nature of the gas with the correct dependences. Only the numerical value of the propagation velocity was (slightly) wrong.