H.P. Paar **PHYS 4B: Mechanics, Fluids, Waves & Heat** Spring 2016

<u>Final solutions</u>

Solutions by Yury Kiselev

1. (10 points)

   (a) (5 points) Using Bernoulli's principle, $P + \rho g z + \rho v^2/2 = \text{const}$, compare this quantity at the top of the water surface and at the exit of the pipe. $v_{top} \approx 0$, because the vessel is large, so

   $$p_0 + \rho g h + 0 = p_0 + 0 + \rho v^2/2.$$

   Then, $v = \sqrt{2gh}$.

   (b) (5 points) Cylinder of water with length $v\Delta t$ will exit the container in time Δt. Mass per second will be

   $$\frac{\Delta M}{\Delta t} = \frac{v \Delta t A \rho}{\Delta t} = \rho v A = \rho A \sqrt{2gh}.$$

2. (25 points)

   (a) (3 points) $PV = NkT$, so $T_A = P_A V_A/(Nk)$, and $T_B = P_B V_B/(Nk)$. Incidentally, $P_A = P_B = P_{AB}$, so
   $T_B - T_A = P_{AB}(V_B - V_A)/(Nk) = T_B - T_A = P_{AB}(V_B - V_A)/(nR)$, and
   $T_B/T_A = (V_B/V_A)$.

   (b) (5 points) The simplest solution, which deserves full credit, is to note that the change is isobaric so is governed by C_P. Thus $dQ = C_P\, dT$ and there is no need to introduce the number of degrees of freedom i as is done in the alternative solution that follows. Integrating gives a solution in terms of T_A and T_B.
   $\delta Q = \mathrm{d}U + P\mathrm{d}V$, where positive heat here means the heat, added to the system.
   $\Delta Q = \int \delta Q$,
   $\int P\mathrm{d}V = P_{AB}(V_B - V_A) = -P_{AB}(V_A - V_B)$,
   $\Delta U = \frac{i}{2} Nk\Delta T$, where i is 3, 5, 7 or other number, depending on the active degrees of freedom of molecules of the gas. So,

   $$\Delta Q = P_{AB}(V_B - V_A) + \frac{i}{2} Nk(T_B - T_A) = P_{AB}(V_B - V_A) + \frac{i}{2} P_{AB}(V_B - V_A) = (\frac{i}{2} + 1) P_{AB}(V_B - V_A).$$

   This is negative heat, as $V_B < V_A$, so the heat is flowing from the system to the environment. Alternatively, $\delta Q = nC_p \mathrm{d}T$ and $Q = nC_p(T_B - T_A) = C_p(V_B - V_A)P_{AB}/R$.

(c) (2 points) See part b), $W = \int P\mathrm{d}V = P_{AB}(V_B - V_A) = -P_{AB}(V_A - V_B)$. The work is negative, so the work is done by the environment on the gas.

(d) (2 points) The work is an area under the graph, with the the sign, determined by whether the volume is decreasing or increasing. Here the volume is decreasing and area is $P_{AB}(V_A - V_B)$, so $W = -P_{AB}(V_A - V_B)$.
Alternatively, the area under the graph is negative because we integrate from the left to the right so $dV < 0$. From that it follows that $V_B < V_A$. It is in principle possible to have an integral whose upper limit is smaller than its lower limit and yet the integral is positive (not the case here).

(e) (5 points) $S_B - S_A = \int_A^B \mathrm{d}S = \int_A^B \frac{\delta Q}{T} = \int_A^B \frac{\mathrm{d}U + P\mathrm{d}V}{T} = \int_A^B \frac{\frac{i}{2}Nk\mathrm{d}T + P\mathrm{d}V}{T}$.
$PV = NkT$, and $P = const$, so $P\mathrm{V} = Nk\mathrm{d}T$, so
$S_B - S_A = \int_A^B \frac{\frac{i}{2}Nk\mathrm{d}T + Nk\mathrm{d}T}{T} = \int_A^B \frac{(1+\frac{i}{2})Nk\mathrm{d}T}{T} = (1 + \frac{i}{2})Nk \ln T_B/T_A =$
$= -(1 + \frac{i}{2})Nk \ln T_A/T_B$.

(f) (2 points) $T_B < T_A$, because $V_B < V_A$ and pressure is constant, so $S_B - S_A$ is negative.

(g) (3 points) The plot should indicate that in the $A \to B$ process the entropy is decreasing and temperature is decreasing as well. $\mathrm{d}S \propto \mathrm{d}T/T$, so $S \propto \ln T + const$, so the curve should resemble a logarithm if T is a horizontal axis. If T is a vertical axis, $T \propto \exp S + const$, so the curve should look like exponent.

(h) (3 points) This is a reversible process, so the change of the entropy is $\mathrm{d}S = \frac{\delta Q}{T}$. The δQ for the gas + environment is zero, because the heat that flows from the gas enters the environment. So, the total change of the entropy is zero. It's true only for reversible processes.
Another answer for full credit is that the second law requires $dS = 0$ for reversible processes and one considers the gas and the environment to be one reversible system.

3. (20 points)

(a) (20 points) In the equilibrium situation, all heat from the heater goes out to the surroundings, so in terms of power, $P_{in} = P_{out}$, so $2\ kW = 50 \cdot (T - 20)\ W$, then $T = 60°C$.

(b) (not graded) When the water is boiling, its temperature is $100°C$, so power lost due to exchange with the surroundings is $P_{out} = 50 \cdot (T - 20)\ W = 4000\ W$. The rest of the heating power goes to evaporating water. So,

$$\Delta t = \frac{Q_v}{P_{in} - P_{out}} = \frac{mL_F}{5\ kW - 4000\ W} = \frac{0.5 \cdot 2.3 \cdot 10^6}{5000 - 4000} = 1150\ s \approx 19\ min.$$

4. (20 points)

(a) (2 points) f is a frequency of oscillations and v_0 is amplitude of velocity oscillations.

(b) (2 points) $T = 1/f$, $\omega = 2\pi f$.

(c) (3 points) Acceleration is a derivative of velocity with respect to time:

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -2\pi f v_0 \sin(2\pi f t).$$

(d) (3 points) To find $x(t)$ we need to integrate, taking into account x_0 — position at $t = 0$:

$$x(t) = x_0 + \int_0^t v(\tau)\,\mathrm{d}\tau = x_0 + \frac{v_0}{2\pi f}\sin(2\pi f\tau)\Big|_0^t = x_0 + \frac{v_0}{2\pi f}\sin(2\pi f t).$$

(e) (3 points)

$$E_k = \frac{mv^2}{2} = \frac{mv_0^2\cos^2(2\pi f t)}{2}.$$

(f) (4 points)

$$E_k^{ave} \equiv \langle E_k \rangle = \frac{mv_0^2}{2}\langle \cos^2(2\pi f t)\rangle.$$

We know that average of sine or cosine squared over time equals to one half, so

$$E_k^{ave} \equiv \langle E_k \rangle = \frac{mv_0^2}{4}.$$

(g) (3 points)

$$x(t) = x_0 + \frac{v_0}{2\pi f}\sin(2\pi f t),$$
$$v(t) = v_0\cos(2\pi f t) = v_0\sin(2\pi f t + \pi/2),$$

so the phase difference between position and velocity is $\pi/2$.

5. (30 points)

(a) (8 points) This is partial differential equation – we have two variables, not one as in ordinary DE. It's also linear (all terms are linear with respect to y and its derivatives), of second order (highest derivative is 2), homogeneous (no constant terms, w.r.t. y and its derivatives), the coefficients are constant (T_s and μ are constants).

(b) (5 points) Two independent solutions, because the order of the equation is two.

(c) (8 points) Solutions are $y = A\sin(kx - \omega t)$ and $y = A\cos(kx - \omega t)$, where we have a constraint, because a velocity of the wave can be found from DE, $v^2 = T_s/\mu$: $\omega/k = v = \sqrt{T_s/\mu}$. Changing the argument to $kx + \omega t$ or other sign changes in the argument do not result in a different solution, that only changes the direction of propagation.

(d) (4 points) The frequency (and angular frequency) is independent from T_s and μ. Amplitude is not fixed too. So, the only constant we can specify is k: $\omega/k = v$, so $k = \omega/v = 2\pi f/v$.

(e) (5 points) Wavelength $\lambda = vT = v/f = \frac{1}{f}\sqrt{T_s/\mu}$; wave number $k = 2\pi f/v$; frequency f is not fixed; $\omega = 2\pi f$ is not fixed either; propagation velocity $v = \sqrt{T_s/\mu}$.

6. (25 points)

(a) (2 points) Volume of the small slab is $\mathrm{d}V = A\mathrm{d}x$, and the mass is $\mathrm{d}m = A\rho_0\mathrm{d}x$.

(b) (5 points) Position of the left side is $x_1^{new} = x + s(x,t)$, and position of the right side is $x_2^{new} = x + \mathrm{d}x + s(x+\mathrm{d}x, t)$.

(c) (5 points) $x_2^{new} = x + \mathrm{d}x + s(x+\mathrm{d}x, t) = x + \mathrm{d}x + s(x,t) + \frac{\partial s}{\partial x}\mathrm{d}x$, so the new volume is $\mathrm{d}V_{new} = A(x_2^{new} - x_1^{new}) = A(1 + \frac{\partial s}{\partial x})\mathrm{d}x$.

(d) (4 points) Mass of the slab does not change, because only position of the atoms/molecules inside the slab changes.

(e) (5 points) Then, $\rho_0 V_1 = \rho V_2$, so $\rho_0 A\mathrm{d}x = \rho A(1 + \frac{\partial s}{\partial x})\mathrm{d}x$ and $\rho = \rho_0/(1 + \frac{\partial s}{\partial x})$.

(f) (4 points) $\frac{\partial s}{\partial x} < 0$ means that the slab is contracting. In this case, temperature will rise in the same way as temperature of ideal gas rises if we decrease volume. Also, we can relate this to friction forces between different layers of the slab.

7. (15 points)
$10\log(I_1/I_2) = B_1 = 20\,\mathrm{d}B$, and $10\log(I_2/I_3) = B_2 = 30\,\mathrm{d}B$, so if we compare I_1 and I_3: $10\log(I_1/I_3) = 10\cdot\log(I_1 I_2/I_3/I_2) = 10(\log(I_1/I_2) + \log(I_2/I_3)) = B_1 + B_2 = 20 + 30 = 50\,\mathrm{d}B$.

8. (25 points)

(a) (4 points) Second beat is emitted after time $T = 1/f$, so the source moved a distance v_s/f towards observer and a new distance $L_2 = L - v_s/f$.

(b) (4 points) Light velocity is equal to c, so $t_2 = L_2/c = (L - v_s/f)/c$.

(c) (4 points) For third beat additional period passed and $L_3 = L - 2v_s/f$ and $t_3 = L_3/c = (L - 2v_s/f)/c$.

(d) (4 points) Time between detection of beats is $\Delta t = t_3 - t_2 + T = 1/f - v_s/(fc)$.

(e) (3 points) Frequency is a period inversed: $f_{det} = 1/\Delta t = \frac{1}{1/f - v_s/(fc)} = \frac{f}{1 - v_s/c}$.

(f) (3 points) $1.5f = \frac{f}{1 - v_s/c}$, so $1 - v_s/c = 2/3$ and $v_s = c/3$, a third of a light speed.

(g) (3 points) It moves towards the observer, because the detected frequency is bigger than emitted one. Looking at the answer for part e), we see that v_s must be positive, so it means that the direction of the velocity coincides with the the velocity indicated in the picture (and not opposite to it), so the galaxy moves towards the obeserver.

9. (20 points)

(a) (2 points) $N-$is a number of elementary constituents of the gas (atoms/molecules).

(b) (7 points) $PV = NkT$ is a universal formula. Due to equipartition theorem, if we take into account three degrees of freedom corresponding to 3D motion, $3kT/2 = \langle mv^2/2\rangle$, then $kT = m/3\langle v^2\rangle$ and $PV = Nm\langle v^2\rangle/3$ is a universal formula too. [We could also prove that formula by calculating pressure]

(c) (4 points) No, it's not correct. Imagine two particles: one has velocity $-u$, another has velocity u. Then, $\langle v^2\rangle = (u^2+u^2)/2 = u^2$, but $\langle v\rangle^2 = [(-u+u)/2]^2 = 0^2 = 0$.

(d) (4 points) As we discussed in part (b), $kT = m\langle v^2\rangle/3 = mv_{rms}^2/3$, so $v_{rms} = \sqrt{3kT/m}$.

(e) (3 points) Again, from (b), $3kT/2 = \langle mv^2/2\rangle = \langle E_k^{translat.}\rangle$.

10. (20 points)
Solids are made of atoms/molecules packed together. We can imagine that some invisible springs connect them with each other. So, atoms can vibrate relative to each other, and this is the only possible degree of freedom for them. At high temperatures vibrations are independent from each other (temperature is high, so the motion is very randomized), so each atom can vibrate in three directions (dimensionality of the space), so contribution to the energy will come from stretching of the springs and from their velocities. In total, we have six degrees of freedom, each of them corresponding to $R/2$, so the total is $3R$. It's the same for all solids because all of them are three-dimensional. There is no surprise that we have R in the expression – it's closely connected to k, which is a universal constant in $e^{-E/(kT)}$ expression, valid for any system.

11. (30 points)

(a) (5 points) We know that the probability involves the Boltzmann Factor multiplied by the differential(s) of the independent variable(s). We only include those differentials whose variables appear in the integrand, v_x in this case. If you include the differentials of the other variables they cancel between numerator and denominator, see below, because the integrands do not depend upon them so they can be put in front of integrals. This leaves $\exp(-\frac{1}{2}mv_x^2/(kT))\,\mathrm{d}v_x$. To get the requested probability we multiply these two factors divide by the integral of these two factors in the denominator. Thus we get

$$\mathrm{d}P(v_x) = \frac{\mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}{\int_{-\infty}^{+\infty} \mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}$$

Because the numerator is infinitely small we write $\mathrm{d}P$ instead of P.

(b) (5 points) To get the average of v_x we use the probability for v_x to be in the interval $[v_x, v_x + \mathrm{d}v_x]$ from a) and multiply it by v_x, the quantity whose average we want, and integrate over all v_x. We get

$$\langle v_x \rangle = \int_{-\infty}^{+\infty} \mathrm{d}P(v_x)\, v_x = \frac{\int_{-\infty}^{+\infty} v_x\, \mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}{\int_{-\infty}^{+\infty} \mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}$$

We have odd function of v_x under integral in the numerator, the interval of the integration is symmetric, so the result is zero because the numerator is not zero.

(c) (5 points)

$$\langle v_x^2 \rangle = \int_{-\infty}^{+\infty} \mathrm{d}P(v_x)\, v_x^2 = \frac{\int_{-\infty}^{+\infty} v_x^2\, \mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}{\int_{-\infty}^{+\infty} \mathrm{e}^{-mv_x^2/(2kT)}\mathrm{d}v_x}$$

(d) (+20 points) $\int_{-\infty}^{+\infty} e^{-\alpha x^2}\, \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$, so after taking derivative of both sides with respect to α, we get

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}(-x^2)\, \mathrm{d}x = (-0.5)\frac{\sqrt{\pi}}{\alpha^{3/2}}.$$

Our expression differs by sign only, so:

$$\int_{-\infty}^{+\infty} v_x^2 \sqrt{\frac{m}{\pi 2kT}} e^{-mv_x^2/(2kT)} = \sqrt{\frac{m}{\pi 2kT}} \cdot (0.5)\frac{\sqrt{\pi}}{(m/(2kT))^{3/2}} = \frac{2kT \cdot 0.5}{m} = \frac{kT}{m}.$$

(e) (5 points) $\langle v_x^2 \rangle$ is bigger than zero, while $\langle v_x \rangle = 0$, so $\langle v_x \rangle^2 = 0$ too and these two expressions are not equal. Also, square of an average of the quantity, should not be the same as average of the squared quantity.

(f) (5 points) We expect it to be equal to $kT/2$, as it corresponds to one degree of freedom.

(g) (+3 point) As we expected, $\frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{2}m \cdot \frac{kT}{m} = kT/2$.

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12. (25 points)

(a) (5 points) Because the expansion is neglected the work $p\,\mathrm{d}V = 0$.

(b) (10 points) $T\mathrm{d}S = \mathrm{d}Q = \mathrm{d}U + p\,\mathrm{d}V = \mathrm{d}U$. The specific heat is $nc = na/T$ so $\mathrm{d}U = na\,\mathrm{d}T/T$. The change in entropy is

$$\Delta S = \int_{T_1}^{T_2} \frac{\delta Q}{T} = \int_{T_1}^{T_2} \frac{\mathrm{d}U}{T} = \int_{T_1}^{T_2} \frac{na}{T}\frac{\mathrm{d}T}{T} = -na\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = na\frac{T_2 - T_1}{T_1 T_2}.$$

This is positive.

(c) (10 points) Make the T, S plot with S – ordinate and T as the abscissa using the result from b). $S = -na/T + na/T_1$, so the graph goes from zero into the positive direction and asymptotically approaches na/T_1.

13. (40 points)

(a) (10 points) Yes, because entropy is a function of a state. This means that the entropy change does not depend on the path and then this substitution allowed.

(b) (5 points) Part II is an adiabatic process: $\mathrm{d}S = 0$, so $\delta Q = 0$. So, $S_2 - S_1 = 0$.

(c) (10 points) Part III is $T = const$ curve, so it's isothermal process. $S_3 - S_2 = \displaystyle\int \frac{\delta Q}{T} = \frac{1}{T}\int \delta Q$. Here $\delta Q = \mathrm{d}U + P\mathrm{d}V$, where positive heat here means the heat, added to the system. $\Delta Q = \displaystyle\int \delta Q$, $\mathrm{d}U = \frac{i}{2}Nk\mathrm{d}T = nC_V\mathrm{d}T = 0$. So,

$$\Delta Q = NkT\ln(V_3/V_2),$$

so $S_3 - S_2 = \displaystyle\int \frac{\delta Q}{T} = Nk\ln(V_3/V_2)$.

(d) (5 points) $S_f - S_i = S_3 - S_2 = Nk\ln(V_3/V_2)$.

(e) (5 points) Yes, as we have seen, $\delta Q > 0$, because $S_3 > S_2$, so the heat flows from the environment to the gas during segment III.

(f) (5 points) $\delta Q = T\mathrm{d}S$, so $Q = \displaystyle\int T\mathrm{d}S$, this is an area between a graph and S-axis in $T - S$ coordinates.

14. (30 points)

(a) (15 points) The probability to find the oscillator in a state n is $p_n = c \cdot e^{-E_n/kT}$, where $E_n = \hbar\omega_0 n$ and c – normalization constant. Sum for all n should give us $p_{sum} = 1$, so $\frac{1}{c} = \displaystyle\sum_{n=0}^{n=\infty} e^{-E_n/kT} = 1 + x + x^2 + ...$, where $x = e^{-\hbar\omega_0/(kT)}$. Using mathematical identity for $|x| < 1$: $(1 + x + x^2 + ...) = (1 - x)^{-1}$, we get $c = 1 - e^{-\hbar\omega_0/(kT)}$. Then, the probability to find oscillator in the ground state is $p_0 = c \cdot e^{-E_0/kT} = (1 - e^{-\hbar\omega_0/(kT)}) \cdot 1 = 1 - e^{-\hbar\omega_0/(kT)}$.

(b) (5 points) When $T = 300\ K$, $p_0 \approx 1 - e^{-1.37*10^3/(300*1.38)} \approx 1 - e^{-3.3} \approx 0.963$.

(c) (5 points) When $T = 1\ K$, $p_0 \approx 1 - e^{-1.37*10^3/(1*1.38)} \approx 1 - e^{-993} \approx 1.0$ with very high precision.

(d) (5 points) For the lower temperatures, the quantum oscillator with bigger probability occupies its ground state.